Real Circles Tangent to Three Conics

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Old Questions and New Perspectives

Our work considers a variant of Apollonius' and Steiner's well known geometric problems counting mutually tangent configurations of conic sections. We enumerate the number of circles tangent to three general conics in the complex projective plane and characterize degenerate configurations. Moreover, we employ computational methods to find instances of three conics with large numbers of real mutually tangent circles. Finally we introduce a machine learning model that, given a configuration of three real conics, accurately predicts the number of real mutually tangent circles. The work is based on the preprint `arXiv:2211.06876`.

Background

We work over an affine chart of $\mathbb{P}^2_{\mathbb{C}}$ that we identify with $\mathbb{A}^2_{\mathbb{C}}$. Recall that a *conic* is the set of points $(x, y) \in \mathbb{A}^2_{\mathbb{C}}$ satisfying the equation

$$Q(x, y) = ax^2 + bxy + cy^2 + dx + ey + f = 0$$

for $a, b, c, d, e, f \in \mathbb{C}$ and a circle of radius r centered at $(s, t) \in \mathbb{A}^2_{\mathbb{C}}$ is the set of points $(x, y) \in \mathbb{A}^2_{\mathbb{C}}$ satisfying the equation

$$C(x, y) = (x - s)^2 + (y - t)^2 - r^2 = 0.$$

The conic and circle intersect at four points, counting multiplicity and including points at infinity, so long as Q and C are irreducible and distinct. A point $(x, y) \in \mathbb{A}^2_{\mathbb{C}}$ satisfying $C(x, y) = Q(x, y) = 0$ is a point of tangency if and only if it has multiplicity at least two, or equivalently that the following determinant

$$\det\left(\left[\nabla Q(x, y) \ \nabla C(x, y)\right]\right)$$

vanishes. This allows us to rephrase the tangency problem as the set of solutions of a polynomial system. Let

$$Q_1(x, y) = a_1x^2 + a_2xy + a_3y^2 + a_4x + a_5y + a_6$$

$$Q_2(x, y) = b_1x^2 + b_2xy + b_3y^2 + b_4x + b_5y + b_6$$

$$Q_3(x, y) = c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y + c_6$$

Fix (u_i, v_i) as the point of tangency on Q_i, we can phrase our geometric problem a parametrized polynomial system $F(x; p)$ with 9 variables $x = (u_1, v_1, \ldots, s, t, r) \in \mathbb{C}^9$ – the points of tangency, center, and radius of the circle – 18 and parameters $p = (a_1, \ldots, a_6, b_1, \ldots, c_6) \in \mathbb{C}^{18}$ – the parameters corresponding to each of the conics.

A Count of the Complex Solutions

Homogenizing the system of polynomails $F(x; p)$ with respect to a new variable z, we expect that the nine degree two polynomials have $2^9 = 512$ intersections. This bound is strict as [8] shows there are at most 184 circles tangent to three conics. We show that this bound is attained generically.

Theorem (Complex Tritanngent Circles)

There are 184 complex circles tangent to three general conics in $\mathbb{P}^2_{\mathbb{C}}$.

The Real Discriminant

We characterize the real discriminant of our polynomial system in the sense of [1]. This is a hypersurface in the parameters

$$(a_1, \ldots, a_6, b_1, \ldots, c_6) \cong \mathbb{R}^{18}$$

where the number of real circles tangent to the three real conics changes. By the implicit function theorem, if Q_1, Q_2, Q_3 do not lie in the real discriminant, the real solutions to our polynomial system are locally smooth functions of Q_1, Q_2, Q_3. In particular, this implies that if Q_1, Q_2, Q_3 are not in the real discriminant, then the number of real solutions is locally constant.

Theorem (Real Discriminant)

An arrangement of three conics Q_1, Q_2, Q_3 is degenerate if and only if one of the following holds:

1. Q_1, Q_2, and Q_3 intersect in at least one point;
2. at least one of Q_1, Q_2, or Q_3 is singular;
3. Q_i and Q_j intersect tangentially for some $i \neq j$;
4. There exists a circle C that is tangent to Q_1, Q_2, Q_3 at $(u_1, v_1), (u_2, v_2), (u_3, v_3)$, respectively, and for some i the curvature of C equals the curvature of Q_i at (u_i, v_i) and the normal vectors $\nabla Q_i(u_i, v_i)$ and $\nabla C(u_i, v_i)$ point in the same direction;
5. There is a line tangent to Q_1, Q_2, and Q_3.

Computation

A Linear Program

The roots of a multivariate polynomial system vary continuously with the parameters. We design a linear program to minimize the complex part of a given solution subject to the constraints that no two real solutions become complex and no two complex solution diverge to infinity. Recursively applying this algorithm, we find constructive instances of conics with high numbers (>100) of real tritangent circles.

Machine Learning

We develop a feed-forward neural network that takes in the coefficients defining the three conic sections and predicts the number of real tritangent circles with 96.3% accuracy after 50 epochs.

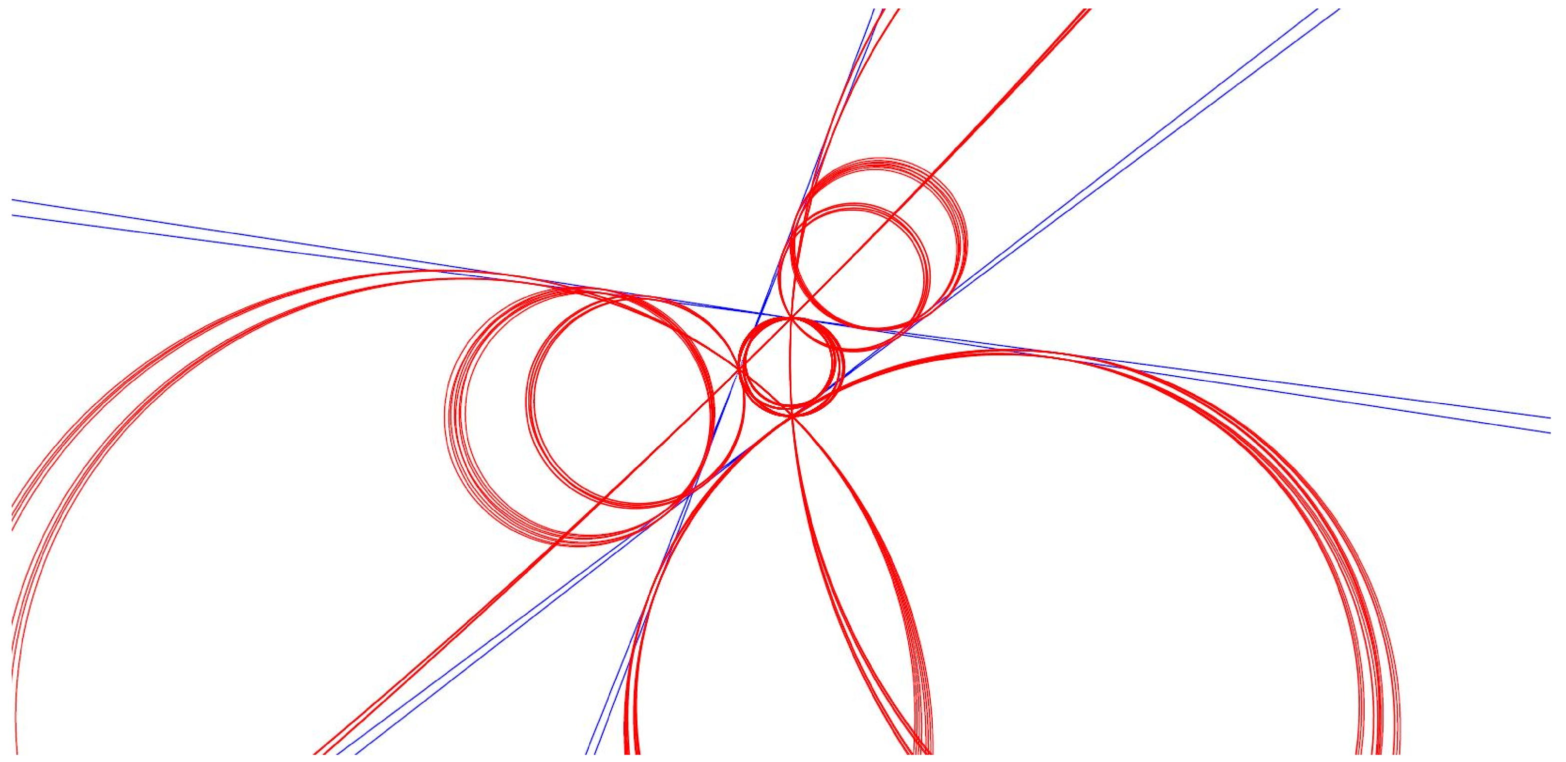

Figure 1. The following instances of three conics (blue) admits 136 real tritangent circles (red).

A Special Configuration

In [10] and [3] exploring the reality of Steiner's Conic Problem, the maximum number of 3264 real conics mutually tangent to five is attained in the neighborhood of a singular configuration of the five conics. Similarly, we would expect that the maximal number of real solutions in the problem of interest would also arise in the neighborhood of a singular configuration.

Theorem

Let Q_1, Q_2, Q_3 be singular conics. There are at most 136 real circles tangent to three conics in a neighborhood of Q_1, Q_2, Q_3.

Our linear program did not find configurations of three conics with more than 136 real tritangent circles. Moreover, the theorem above shows techniques used in [10] cannot be used to show that there exist three conics with 184 real tritangent circles, it does not rule out the possibility that such configurations of conics exist. This being said, we conjecture that 136 is the maximium number of real conics.

Conjecture

There are at most 136 real circles tangent to three general conics in $\mathbb{P}^2_{\mathbb{R}}$.

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