The `A1BrouwerDegrees` Package in `Macaulay2`

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Joint work with N. Borisov, T. Brazelton, F. Espino, T. Hagedorn, Z. Han, J. Lopez Garcia, J. Louwsma, A. Tawfeek *arXiv: 2312:00106*

Conversations with S. McKean, G. Muratore, and S. Pauli

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An Overview

Brouwer Degrees: Ordinary and Motivic

Computing in the Grothendieck-Witt Ring

Enumerative Geometry: Degrees and Quadratic Forms

Brouwer Degrees: Ordinary and Motivic

The Classical Brouwer Degree

Definition (Brouwer Degree)

Let $f : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial map. The Brouwer degree of f is given by

$$\sum_{x \in f^{-1}(0)} \mathrm{ind}_x f = \sum_{x \in f^{-1}(0)} \det\left(\mathrm{Jac}(f)(x)\right).$$

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A tool for classical enumerative geometry:

Theorem (Poincaré-Hopf)

Let V be an vector bundle over a smooth closed oriented manifold X. If σ is any section of V then

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Example

For the 27 lines on the cubic surface, we compute the Euler class of $\mathrm{Sym}^3 \mathcal{S}^\vee$.

Algebraic Geometry is Not Enough

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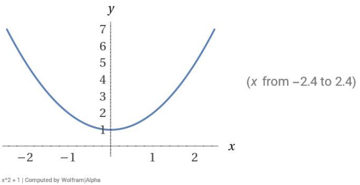
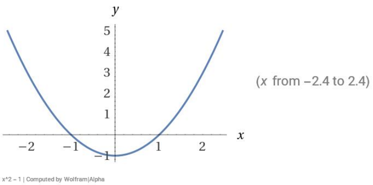
Example

Sections of the line bundle $\mathcal{O}_{\mathbb{A}^1_{\mathbb{R}}}(2)$ tell us about the vanishing locus of quadratic equations.





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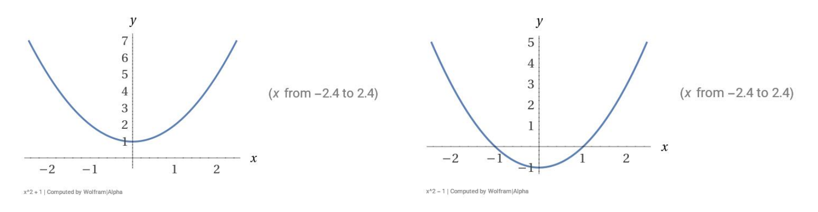

Remark: While the naïve count is not invariant, a *signed* count is.

Working with motivic spaces instead of schemes captures field-specific arithmetic information.

Theorem (Morel)

Let k be a field of characteristic not 2 and $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial map with $n \geqslant 2$. f admits a well-defined degree valued in the Grothendieck-Witt ring $\mathrm{GW}(k)$ of non-degenerate symmetric bilinear forms.

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- Over $k = \mathbb{C}$, the rank of the form recovers the classical Brouwer degree.
- Over other fields, the invariants of the symmetric bilinear form capture field-specific information.

Theorem (Bachmann-Wickelgren, Kass-Wickelgren)

Let V be an oriented algebraic vector bundle over a scheme X smooth and proper over $\mathrm{Spec}(k)$ with $\mathrm{char}(k) \neq 2$. If σ is any section of V then

$$n^{\mathbb{A}^1}(V) = \sum_{x \in \sigma^{-1}(0)} \mathrm{ind}_x^{\mathbb{A}^1} \sigma$$

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Upshot: This recovers Schubert's "invariance of number" over non algebraically closed fields.

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There are explicit commutative-algebraic formulae for computing these $\mathbb{A}^1$-Brouwer degrees.

Theorem (Brazelton-McKean-Pauli)

Let k be a field of characteristic not 2 and $f : \mathbb{A}^n_k \to \mathbb{A}^n_k$ be a polynomial map. The $\mathbb{A}^1$-degree of f can be computed using the multivariate Bézoutian.

Remark: These multivariate Bézoutians can be thought of as a formal Jacobian matrix.

The A1BrouwerDegrees Package

The package *A1BrouwerDegrees* provides software to **perform basic operations on the Grothendieck-Witt ring** $\mathrm{GW}(k)$ and **compute these enriched enumerative counts**.

The package was included in release 1.23 of ***Macaulay2*** and can be imported as follows:

```
i1 : loadPackage "A1BrouwerDegrees"

o1 = A1BrouwerDegrees

o1 : Package
```

Note that a number of method names will be changed in the next release of the package to better adhere to ***Macaulay2*** style conventions.

Computing in the Grothendieck-Witt Ring

Definition (Grothendieck-Witt Ring)

Let k be a field of characteristic not 2. The Grothendieck-Witt ring $\mathrm{GW}(k)$ is the set of isomorphism classes of symmetric bilinear forms under $\oplus$ and $\otimes$ group completed with respect to $\oplus$.

The Grothendieck-Witt ring is additively generated by $\langle a \rangle$ for $a \in k^\times/(k^\times)^2$.

This is equivalent to the fact that every symmetric matrix is congruent to a diagonal matrix.

We define a Grothendieck-Witt class by its underlying Gram matrix with the *gwClass* constructor:

```
i2 : M = matrix(QQ, {{1, 2}, {2, 5}})

o2 = | 1 2 |
     | 2 5 |

                 2        2
o2 : Matrix QQ  <-- QQ

i3 : alpha = gwClass(M)

o3 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 1 2 |
                                     | 2 5 |

o3 : GrothendieckWittClass
```

We can add and multiply classes in the Grothendieck-Witt ring.

```
i4 : N = matrix(QQ, {{7, 3}, {3, 2}});

                 2       2
o4 : Matrix QQ  <-- QQ

i5 : beta = gwClass(N);

i6 : gamma = gwAdd(alpha, beta)

o6 = GrothendieckWittClass{cache => CacheTable{}}
                           matrix => | 1 2 0 0 |
                                     | 2 5 0 0 |
                                     | 0 0 7 3 |
                                     | 0 0 3 2 |

o6 : GrothendieckWittClass
```

Simplifying Representatives

By Sylvester's law of inertia, every symmetric bilinear form is isomorphic to a diagonal one.

```
i7 : P = matrix(GF(19), {{1, 5, 17, 8}, {5, 3, 9, 4},
                   {17, 9, 13, 2}, {8, 4, 2, 6}});

                          4             4
o7 : Matrix (GF 19)  <-- (GF 19)

i8 : delta = gwClass(P);

i9 : diagonalClass(delta)

o9 = GrothendieckWittClass{cache => CacheTable{} }
                           matrix => | 1 0 0  |
                                     | 0 1 0 0  |
                                     | 0 0 1 0  |
                                     | 0 0 0 -1 |

o9 : GrothendieckWittClass
```

Isomorphism of Forms

We can also read off the decomposition of a quadratic form as a sum of hyperbolic forms and its anisotropic part.

```
i10 : sumDecompositionString(delta)

o10 = 1H+ <1>+ <6>
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Note that this is the same as the decomposition as the form with diagonal entries 1, 1, 1, -1 we computed previously since

```
i11 : A = diagonalForm(GF(19), (1, 1, 1, -1));

i12 : B = diagonalForm(GF(19), (1, -1, 1, 6));

i13 : gwIsomorphic(A, B)

o13 = true
```

We can also compute a number of arithmetic invariants of symmetric bilinear forms such as its signature

```
i14 : T = matrix(QQ, {{1, 7, 2}, {7, 9, 3}, {2, 3, 5}});

                 3       3
o14 : Matrix QQ  <-- QQ

i15 : epsilon = gwClass(T);

i15 : signature(epsilon)

o15 = 1
```

and its Hasse-Witt invariant with respect to a prime *p*.

```
i16 : HasseWittInvariant(epsilon, 5)

o16 = 1
```

Enumerative Geometry: Degrees and Quadratic Forms

We can compute the number of lines on a cubic surface by taking sections of $\mathrm{Sym}^3 \mathcal{S}^\vee$.

Theorem (Kass-Wickelgren)

Let k be a field of characteristic not two and $X \subseteq \mathbb{P}^3_k$ a smooth cubic surface. Then X contains $15\langle 1 \rangle + 12\langle -1 \rangle$ lines.

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Remark

Over $\mathbb{R}$ quadratic forms are classified by rank and signature and $15\langle 1\rangle + 12\langle -1\rangle$ has signature 3, recovering a result of Segre that the difference of the number of hyperbolic lines (ie. type $\langle 1\rangle$) and elliptic lines (ie. type $\langle -1\rangle$) is 3.

Computing the Lines on a Smooth Cubic Surface

The $\mathbb{A}^1$-Euler number of $\mathrm{Sym}^3\mathcal{S}^\vee$ is the $\mathbb{A}^1$-degree of sections over an affine patch of $\mathbb{G}_{\mathbb{Q}}(1,3)$. Let $X = V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subseteq \mathbb{P}^3$.

```
i17 : R = QQ[y_1,y_2,y_3,y_4];

i18 : f = {y_1^3 + y_3^3 + 1,
          3*y_1^2*y_2 + 3*y_3^2*y_4,
          3*y_1*y_2^2 + 3*y_3*y_4^2,
          y_2^3 + y_4^3 + 1};

i19 : alpha = globalA1Degree(f);

i20 : sumDecompositionString(alpha)

o20 = 8H+ <1>+ <1>
```

Remark

This is a form of rank 18, reflecting the number of lines defined by points **on this chart** of $\mathbb{G}_{\mathbb{Q}}(1,3)$. Summing over different charts will yield the desired rank 27 form.

We can compute a primary decomposition of the ideal defined by the equations of f and consider the *local* $\mathbb{A}^1$*-degree* of f at this point.

```
i21 : I = (minimalPrimes ideal f)_0

o21 = ideal (y , y  + 1, y  + 1, y )
              4   3       2       1

o21 : Ideal of R

i22 : beta = localA1Degree(f, I);

i23 : sumDecompositionString beta

o23 = <1>
```

From work of Kass-Wickelgren, the local index of a line on a cubic surface is determined by a certain resultant.

```
i11 : S = QQ[z_1, z_2][z_3, z_4];

i12 : F = (z_1 + z_4)^3 + (z_2 + z_3)^3 - z_3^3 - z_4^3;

i13 : g1 = sub(diff(z_1, F), {z_1 => 0, z_2 => 0});

i14 : g2 = sub(diff(z_2, F), {z_1 => 0, z_2 => 0});

i16 : line_type = diagonalForm(QQ, lift(resultant {g1,g2}, QQ))

o16 = GrothendieckWittClass{cache => CacheTable{}}
                            matrix => | 81 |

o16 : GrothendieckWittClass

i17 : gwIsomorphic(line_type, beta)

o17 = true
```

Takeaways

- $\mathbb{A}^1$-enumerative geometry as an enhancement of classical enumerative geometry over $\mathbb{C}$.

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- Future improvements:
    - Traces when the preimage is not k-rational.
    - Unstable $\mathbb{A}^1$-degrees, tropical enumerative geometry, ...

Thank You. Questions?

We consider the A_2 cuspidal singularity defined by $V(x^2 + y^3) \subseteq \mathbb{A}^2_{\mathbb{F}_{32003}}$.

```
i18 : S = GF(32003)[x,y];

i19 : G = x^2 + y^3;

i20 : beta = globalA1Degree({diff(x, G), diff(y, G)});

i21 : sumDecompositionString(beta)

o21 = 1H
```

The rank of this form recovers the classical Milnor number of the cusp, and records that the the cusp bifurcates into two nodes.

On the other hand, we can compute the $\mathbb{A}^1$-Milnor number of a node and show that the cusp is not isomorphic to the node.

```
i22 : H = x^2 + y^2;

i23 : nodeDeg = globalA1Degree({diff(x, H), diff(y, H)})

o23 = GrothendieckWittClass{cache => CacheTable{}}
                            matrix => | 4 |

o23 : GrothendieckWittClass

i24 : gwIsomorphic(nodeDeg, hyperbolicForm(GF(32003)))

o24 = false
```

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