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Preliminaries

These are the author's notes for the third talk at the inaugural Kleine AT conference on the construction of spectral sequences from filtered objects via décalage.

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1. Recollections

Recall the following from the first talk:

- Let $\mathcal{C}$ be a stable ∞-category. This is the appropriate analogue of an Abelian category in the higher setting (eg. fibers and cofibers exist).
- We will assume that $\mathcal{C}$ has sequential limits and colimits.
- For a decreasing filtration $F^\star \in \mathsf{F}(\mathcal{C}) \simeq \mathrm{Fun}(\mathbb{Z}^{\mathsf{Opp}}, \mathcal{C})$, we denote $|F^\star| \simeq \operatorname{colim}_s F^{-s}$ its underlying object.
- The category $\mathsf{F}(\mathcal{C})$ of decreasing filtrations in $\mathcal{C}$ is stable and can be endowed with the Beilinson t-structure.

2. Spectral Sequences in General

We first consider the simple case of a spectral sequence defined on an Abelian 1-category.

Definition 2.1 (Spectral Sequence – Abelian Category)**.** Let A be an Abelian category. A spectral sequence in A starting at the ath page consists of:

(i) Objects $E^r_{s,t}$ of A for $r \geq a$ and $s, t \in \mathbb{Z}$.
(ii) Differentials $d^r_{s,t} : E^r_{s,t} \to E^r_{s-r,t+r-1}$ for $r \geq a$ and $s, t \in \mathbb{Z}$.
(iii) Isomorphisms between the cohomology at $E^r_{s,t}$ with respect to d^r and $E^{r+1}_{s,t}$ for all $r \geq a$ and $s, t \in \mathbb{Z}$.

We will by convention assume that spectral sequences start at page 1. So the E^1-page of a spectral sequence is given by

$$\begin{tikzcd}
E^1_{-2,1} & E^1_{-1,1} \arrow[l, "d^1_{-1,1}"'] & E^1_{0,1} \arrow[l, "d^1_{0,1}"'] & E^1_{1,1} \arrow[l, "d^1_{1,1}"'] & E^1_{2,1} \arrow[l, "d^1_{2,1}"'] \\
E^1_{-2,0} & E^1_{-1,0} \arrow[l, "d^1_{-1,0}"'] & E^1_{0,0} \arrow[l, "d^1_{0,0}"'] & E^1_{1,0} \arrow[l, "d^1_{1,0}"'] & E^1_{2,0} \arrow[l, "d^1_{2,0}"'] \\
E^1_{-2,-1} & E^1_{-1,-1} \arrow[l, "d^1_{-1,-1}"'] & E^1_{0,-1} \arrow[l, "d^1_{0,-1}"'] & E^1_{1,-1} \arrow[l, "d^1_{1,-1}"'] & E^1_{2,-1} \arrow[l, "d^1_{2,-1}"']
\end{tikzcd}$$

FIGURE 1. E^1-page of a spectral sequence.

where all the rows are chain complexes in A. We take the homology at each entry to get to the E^2-page which is given by

$$\begin{tikzcd}
E^2_{-2,1} & E^2_{-1,1} & E^2_{0,1} & E^2_{1,1} & E^2_{2,1} \\
E^2_{-2,0} & E^2_{-1,0} & E^2_{0,0} \arrow[ull, "d^2_{0,0}" description] & E^2_{1,0} \arrow[ull, "d^2_{1,0}" description] & E^2_{2,0} \arrow[ull, "d^2_{2,0}" description] \\
E^2_{-2,-1} & E^1_{-1,-1} & E^2_{0,-1} \arrow[ull, "d^2_{0,-1}" description] & E^2_{1,-1} \arrow[ull, "d^2_{1,-1}" description] & E^2_{2,-1} \arrow[ull, "d^2_{2,-1}" description]
\end{tikzcd}$$

FIGURE 2. E^2-page of a spectral sequence.

where we have implicitly supplied isomorphisms,

$$E^2_{s,t} \cong \frac{\ker\left(d^1_{s,t} : E^1_{s,t} \to E^1_{s-1,t}\right)}{\operatorname{im}\left(d^1_{s+1,t} : E^1_{s+1,t} \to E^1_{s,t}\right)}.$$

Spectral sequences are an important computational tool: for a difficult-to-understand object X, we can populate $E^a_{s,t}$ with some computable invariants of X and hope to understand X better from the terms of $E^r_{s,t}$. This is illustrated by the following example.

Example 2.2 (Atiyah-Hirzebruch Spectral Sequence)**.** Let $\mathcal{H}$ be a nice cohomology theory and X a CW complex. The Atiyah-Hirzebruch spectral sequence is the spectral sequence starting on the E^2-page with

$$E^2_{s,t} = H^s(X, \mathcal{H}^t(\{*\})).$$

It is a fact that the Atiyah-Hirzebruch spectral sequence stabilizes for sufficiently large r (ie. the E^r and E^{r+1}-pages agree for $r >>> 0$), and it is in this sense that a spectral sequence can be said to converge. The so-called E^∞-page produces the associated grades for a filtration on $\mathcal{H}^{s+t}(X)$ along the diagonals. The terms of the E^2-page are easily obtained by the universal coefficient theorem.

More generally, for a stable ∞-category, we know that its heart $\mathcal{C}^\heartsuit$ is an Abelian 1-category, and Definition 2.1 generalizes immediately to $\mathcal{C}$ in the obvious way.

Definition 2.3 (Spectral Sequence – Stable Category)**.** Let $\mathcal{C}$ be a stable ∞-category. A spectral sequence in $\mathcal{C}$ starting at the ath page consists of:

(i) Objects $E^r_{s,t}$ of $\mathcal{C}^\heartsuit$ for $r \geq a$ and $s, t \in \mathbb{Z}$.
(ii) Differentials $d^r_{s,t} : E^r_{s,t} \to E^r_{s-r,t+r-1}$ for $r \geq a$ and $s, t \in \mathbb{Z}$.
(iii) Isomorphisms between the cohomology at $E^r_{s,t}$ with respect to d^r and $E^{r+1}_{s,t}$ for all $r \geq a$ and $s, t \in \mathbb{Z}$.

3. Spectral Sequence of a Filtered Object

One way that spectral sequences arises is from filtered objects, the desideratum here being that we want to make conclusions about the homotopy objects $\pi_*|F|$ of the underlying object $|F|$ for $F \in \mathsf{F}(\mathcal{C})$ from the associated graded objects which are more easily acccessible.

Observe here that a spectral sequence (which we have assumed to start at the E^1-page) has chain complexes along the horizontals with differentials pointing left.

Recalling the result from the first talk that the composition $\mathrm{gr}^s_F \to \mathrm{gr}^{s+1}_F[1] \to \mathrm{gr}^{s+2}_F[2]$ is canonically nullhomotopic [1, Lem. 3.18], allows us to think of the sequential diagram

$$\cdots \to \mathrm{gr}^{-s-1}_F[-s-1] \to \mathrm{gr}^{-s}_F[-s] \to \mathrm{gr}^{-s+1}_F[-s+1] \to \dots \tag{3.1}$$

as a cochain complex in $\mathcal{C}$. After passing to homotopy objects – and appealing to commutativity of the formation of homotopy objects with the shift – we have that

$$\cdots \to \pi_{s+t+1}\mathrm{gr}^{-s-1}_F \to \pi_{s+t}\mathrm{gr}^{-s}_F \to \pi_{s+t-1}\mathrm{gr}^{-s+1}_F \to \dots \tag{3.2}$$

is an honest cochain complex in $\mathcal{C}^\heartsuit$. The cochain complexes (3.2) form natural candidates for the horizonal entries of a spectral sequence that we could associate to F.

We will show this expectation indeed holds.

Proposition 3.1. Let $\mathcal{C}$ be a stable ∞-category and $F^\star$ decreasing filtration in $\mathcal{C}$. Define for $r \geq 1$ the collection of objects

$$E^r_{s,t} = \mathrm{im}\left(\pi_{s+t}\mathrm{gr}^{[-s,-s+r)}_F \to \pi_{s+t}\mathrm{gr}^{[-s-r+1,-s+1)}_F\right) \in \mathcal{C}^\heartsuit.$$

There exist differentials $d^r_{s,t} : E^r_{s,t} \to E^r_{s-r,t+r-1}$ and isomorphisms between the cohomology at $E^r_{s,t}$ and $E^{r+1}_{s,t}$ such that the objects, together with the differentials and isomorphisms, define a spectral sequence in the sense of Definition 2.3 starting on the E^1-page. In particular, the E^1-page of this spectral sequence is given by $E^1_{s,t} = \pi_{s+t}\mathrm{gr}^{-s}_F$, and the differentials agree with those in (3.2).

Proof. For the first part, we only show that the maps $d^r_{s,t} : E^r_{s,t} \to E^r_{s-r,t+r-1}$ exist. That these maps are indeed differentials and the isomorphism from homology to the term on the successive page will follow from the comparison that arises from the décalage construction.

Using the commuting diagram

$$\begin{tikzcd}
F^{-s+2r} \arrow[r] \arrow[d] & F^{-s+r} \arrow[r] \arrow[d] & F^{-s} \arrow[d] \\
F^{-s+r+1} \arrow[r] & F^{-s+1} \arrow[r] & F^{-s-r+1}
\end{tikzcd}$$

we obtain by an iterated taking of cofibers the diagram

$$\begin{tikzcd}
\mathrm{gr}_F^{[-s+r,-s+2r)} \arrow[r] \arrow[d] & \mathrm{gr}_F^{[-s,-s+2r)} \arrow[r] \arrow[d] & \mathrm{gr}_F^{[-s,-s+r)} \arrow[d] \\
\mathrm{gr}_F^{[-s+1,-s+r+1)} \arrow[r] & \mathrm{gr}_F^{[-s-r+1,-s+r+1)} \arrow[r] & \mathrm{gr}_F^{[-s-r+1,-s+1)}
\end{tikzcd}$$

with rows cofiber sequences. Passing to homotopy objects gives the commutative diagram of dotted arrows

$$\begin{tikzcd}
\cdots \arrow[r, dotted] & \pi_{s+t}\mathrm{gr}_F^{[-s,-s+r)} \arrow[rrr, dotted] \arrow[dd, dotted] \arrow[dr] & & & \pi_{s+t-1}\mathrm{gr}_F^{[-s+r,-s+2r)} \arrow[r, dotted] \arrow[dd, dotted] \arrow[dl] & \cdots \\
& & E^r_{s,t} \arrow[r, dashed, "\exists!"] \arrow[dl] & E^r_{s-r,t+r-1} \arrow[dr] & & \\
\cdots \arrow[r, dotted] & \pi_{s+t}\mathrm{gr}_F^{[-s-r+1,-s+1)} \arrow[rrr, dotted] & & & \pi_{s+t-1}\mathrm{gr}_F^{[-s+1,-s+r+1)} \arrow[r, dotted] & \cdots
\end{tikzcd}$$

and the rows are the long exact sequence for homotopy objects of a fiber sequence. The solid arrows exist by factorization of the vertical maps into an epimorphism followed by a monomorphism, and the dashed arrow exists uniquely by functoriality of such factorizations.

For the second statement, setting $r = 1$, we have that

$$\begin{aligned}
E^1_{s,t} &\cong \mathrm{im}\left(\pi_{s+t}\mathrm{gr}_F^{[-s,-s+1)} \to \pi_{s+t}\mathrm{gr}_F^{[-s,s+1)}\right) \\
&\cong \mathrm{im}\left(\pi_{s+t}\mathrm{gr}_F^{-s} \xrightarrow{\mathrm{id}_{\pi_{s+t}\mathrm{gr}_F^{-s}}} \pi_{s+t}\mathrm{gr}_F^{-s}\right) \\
&\cong \pi_{s+t}\mathrm{gr}_F^{-s}
\end{aligned}$$

where in the middle we have used that $\mathrm{gr}_F^{[-s,-s+1)} \simeq \mathrm{cofib}(F^{-s+1} \to F^{-s}) \simeq \mathrm{gr}_F^i$ and that the map is induced along the identity. Compatibility with the differentials is clear as unwinding the preceding constructions for $r = 1$ we recover precisely the map $\delta : \mathrm{gr}_F^{-s} \to \mathrm{gr}_F^{-s+1}[1]$ of the previous talk (cf. [1, Lem. 3.18]). ■

Definition 3.2 (Lurie – Spectral Sequence of Filtered Object)**.** Let $\mathcal{C}$ be a stable ∞-category and $F^\star$ decreasing filtration in $\mathcal{C}$. The (Lurie) spectral sequence $E^r_{s,t}(F)$ of $F^\star$ is the spectral sequence of Proposition 3.1.

$$\begin{array}{ccccccccc}
\pi_{-1}\mathrm{gr}^2_F & \longleftarrow & \pi_0\mathrm{gr}^1_F & \longleftarrow & \pi_1\mathrm{gr}^0_F & \longleftarrow & \pi_2\mathrm{gr}^{-1}_F & \longleftarrow & \pi_3\mathrm{gr}^{-2}_F \\
\\
\pi_{-2}\mathrm{gr}^2_F & \longleftarrow & \pi_{-1}\mathrm{gr}^1_F & \longleftarrow & \pi_0\mathrm{gr}^0_F & \longleftarrow & \pi_1\mathrm{gr}^{-1}_F & \longleftarrow & \pi_2\mathrm{gr}^{-2}_F \\
\\
\pi_{-3}\mathrm{gr}^2_F & \longleftarrow & \pi_{-2}\mathrm{gr}^1_F & \longleftarrow & \pi_{-1}\mathrm{gr}^0_F & \longleftarrow & \pi_0\mathrm{gr}^{-1}_F & \longleftarrow & \pi_1\mathrm{gr}^{-2}_F
\end{array}$$

FIGURE 3. E^1-page of the spectral sequence of a filtered object.

It is easy to see that the t-th row of $E^1_{s,t}(F)$ is precisely the cochain complex (3.2).

4. Décalage

The décalage construction produces a new filtered object $\mathrm{Déc}(F) \in \mathsf{F}(\mathcal{C})$ in a natural way from the Beilinson t-structure on $\mathcal{C}$.

Recall that formation of Whitehead towers $\tau^B_{\geq n} : \mathsf{F}(\mathcal{C}) \to \mathsf{F}(\mathcal{C})_{\geq n}$ is right adjoint to the full inclusion of $\mathsf{F}(\mathcal{C})_{\geq n} \subseteq \mathsf{F}(\mathcal{C})$. This furnishes natural maps $\tau^B_{\geq n}F \to F^\star$ and more generally $\tau^B_{\geq n+1}F \to \tau^B_{\geq n}F$ in $\mathsf{F}(\mathcal{C})$ which compose to the sequential diagram

$$\cdots \to \tau^B_{\geq n+1}F \to \tau^B_{\geq n}F \to \tau^B_{\geq n-1}F \to \dots \tag{4.1}$$

and the décalage of F is defined to be the filtered object obtained by passing to the realization of (4.1).

Definition 4.1 (Décalage)**.** Let $\mathcal{C}$ be a stable ∞-category and $F^\star$ a decreasing filtration in $\mathcal{C}$. The décalage $\mathrm{Déc}(F)^\star$ of F is the decreasing filtration in $\mathcal{C}$ given by $\mathrm{Déc}(F)^i = |\tau^B_{\geq i}F|$.

Here, we have treated each Whitehead tower $\tau^B_{\geq n}F$ as a filtered object, and considered its underlying object.

Before going further, let us verify record elementary properties of the décalage.

Lemma 4.2. Let $\mathcal{C}$ be a stable ∞-category and $F^\star$ a decreasing filtration in $\mathcal{C}$.

(i) There is an equivalence on underlying objects $|F^\star| \simeq |\mathrm{Déc}(F)^\star|$.

(ii) The associated graded $\mathrm{gr}^n_{\mathrm{Déc}(F)}$ is given by $|\pi^B_n F|[n] \simeq |(\pi^B_n F)[n]|$, where $\pi^B_n F$ is the cochain complex

$$\cdots \to \pi_{n+1}\mathrm{gr}^{-1}_F \to \pi_n\mathrm{gr}^0_F \to \pi_{n-1}\mathrm{gr}^1_F \to \dots$$

in $\mathcal{C}^\heartsuit$ where $\pi_n\mathrm{gr}^0_F$ is in cohomological degree 0.

Proof of (i). This is clear from $\tau^B_{\geq -\infty}F \simeq F^\star$. ∎

Proof of (ii). We compute

$$\begin{aligned}\mathrm{gr}^n_{\mathrm{Déc}(F)} &\simeq \mathrm{cofib}\left(\left|\tau^B_{\geq n+1}F\right| \to \left|\tau^B_{\geq n}F\right|\right)\\ &\simeq \left|\mathrm{cofib}\left(\tau^B_{\geq n+1}F \to \tau^B_{\geq n}F\right)\right|\\ &\simeq \left|(\pi^B_n F)[n]\right|\end{aligned}$$

and using that the shift is computed pointwise gives the first statement.

For the second statement, we use that $\pi^B_n F \in \mathsf{F}(\mathcal{C})^\heartsuit$ so we can use the equivalence $\mathrm{Ch}^\bullet(\mathcal{C}^\heartsuit) \simeq \mathsf{F}(\mathcal{C})^\heartsuit$ to identify $\pi^B_n F$ with the cochain complex $C^\bullet$ with $C^i = \mathrm{gr}^i_{\pi^B_n F}[i]$. By t-exactness of the associated graded, we have that $\mathrm{gr}^i_{\pi^B_n F}[i] \simeq \pi_{n-i}\mathrm{gr}^i_F$. See [2, Prop. 5.8 & Corr. 5.9] for more details. ■

Denote $\mathrm{Déc}^{(r)}(F)$ to be the r-fold iterate of the décalage functor, taking $\mathrm{Déc}^{(0)}(F)$ to be the identity. The preceding lemma shows that the underlying objects $|F^\star|$ and $|\mathrm{Déc}^{(r)}(F)^\star|$ agree.

Lemma 4.3. Let $\mathcal{C}$ be a stable ∞-category and $F^\star$ a decreasing filtration in $\mathcal{C}$. Define for $r \geq 0$ the collection of objects

$$E^{r+1}_{s,t} = \pi_{s+t}\mathrm{gr}^{rs+rt-s}_{\mathrm{Déc}^{(r)}(F)}.$$

There exist differentials $d^r_{s,t} : E^r_{s,t} \to E^r_{s-r,t+r-1}$ and isomorphisms between the cohomology at $E^r_{s,t}$ and $E^{r+1}_{s,t}$ such that the objects, together with the differentials and isomorphisms, define a spectral sequence in the sense of Definition 2.3 starting on the E^1-page. In particular, the E^1-page of this spectral sequence is given by $E^1_{s,t} = \pi_{s+t}\mathrm{gr}^{-s}_F$, and the differentials agree with those in (3.2).

Proof. The second statement just follows formally from the fact that the 0-fold composite of the décalage functor is the identity on F.

By induction, it suffices to consider the case of $r = 1$. Here we have maps $d^2_{s,t} : \pi_{s+t}\mathrm{gr}^t_{\mathrm{Déc}(F)} \to \pi_{s+t-1}\mathrm{gr}^{t-1}_{\mathrm{Déc}(F)}$ but we already know these are differentials. The homology at $E^1_{s,t}$ is given by

$$\begin{aligned}\frac{\ker\left(\pi_{s+t}\mathrm{gr}^{-s}_F \to \pi_{s+t-1}\mathrm{gr}^{-s+1}_F\right)}{\mathrm{im}\left(\pi_{s+t+1}\mathrm{gr}^{-s-1}_F \to \pi_{s+t}\mathrm{gr}^{-s}_F\right)} &= H^{-s}(\pi^B_t F)\\ &\cong \pi_s|\pi^B_t F|\\ &= \pi_{s+t}|\pi^B_t F|[t]\\ &= \pi_{s+t}\mathrm{gr}^t_{\mathrm{Déc}(F)} = E^2_{s,t}\end{aligned}$$

where the first equality is by the definition of $\pi^B_t F$ as the cochain complex

$$\begin{array}{ccccccccc} & & -s+1 & & -s & & -s-1 & & \\ \dots & \longrightarrow & \pi_{s+t+1}\mathrm{gr}^{-s-1}_F & \longrightarrow & \pi_{s+t}\mathrm{gr}^{-s}_F & \longrightarrow & \pi_{s+t-1}\mathrm{gr}^{-s+1}_F & \longrightarrow & \dots\end{array}$$

after reindexing, and the second canonical isomorphism provided by $\pi_s|C^\bullet| \cong H^{-s}(C^\bullet)$ for cochain complexes $C^\bullet$. ■

We can then define this as a spectral sequence.

Definition 4.4 (Décalage – Spectral Sequence of Filtered Object)**.** Let $\mathcal{C}$ be a stable ∞-category and $F^\star$ decreasing filtration in $\mathcal{C}$. The décalage spectral sequence $E^r_{s,t}(F)$ of $F^\star$ is the spectral sequence of Lemma 4.3.

5. Towards the Comparison Result

We have thus far produced two constructions for a spectral sequence associated to $F^\star \in \mathsf{F}(\mathcal{C})$

- (Lurie) $E^r_{s,t} = \operatorname{im}\left(\pi_{s+t}\mathrm{gr}_F^{[-s,-s+r)} \to \pi_{s+t}\mathrm{gr}_F^{[-s-r+1,-s+1)}\right)$, and
- (Décalage) $E^r_{s,t} = \pi_{s+t}\mathrm{gr}^{rs+rt-2s-t}_{\mathrm{Déc}^{(r-1)}(F)}$.

We know that the E^1-pages of these two constructions agree, and by definition of a spectral sequence that the higher terms of the pages of the respective spectral sequences are abstractly isomorphic, albeit not necessarily in a way compatible with the differentials.

However, it turns out that these definitions are equivalent, and that taking décalage computes the subsequent page of the spectral sequence (modulo some index shifting). We conclude by stating the result that will be shown in the following talk.

Theorem 5.1. Let $\mathcal{C}$ be a stable ∞-category and $F^\star$ decreasing filtration in $\mathcal{C}$. For each $r \geq 1$ there are natural isomorphisms $E^r_{-t,s+2t}(\mathrm{Déc}(F)) \cong E^{r+1}_{s,t}(F)$ compatible with the differentials.

References

[1] Benjamin Antieau. *Spectral sequences, décalage, and the Beilinson t-structure*. 2024. arXiv: `2411.09115 [math.AT]`. URL: `https://arxiv.org/abs/2411.09115`.

[2] Stefano Ariotta. *Coherent cochain complexes and Beilinson t-structures, with an appendix by Achim Krause*. 2021. arXiv: `2109.01017 [math.AT]`. URL: `https://arxiv.org/abs/2109.01017`.

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