V5A2 – THE HABIRO RING OF A NUMBER FIELD
WINTER SEMESTER 2024/25

WERN JUIN GABRIEL ONG

Dedication

The course is dedicated to the memory of Tobias Kreutz, a postdoctoral fellow at the Max Planck Institute for Mathematics, who passed away in August 2024.

Preliminaries

These notes roughly correspond to the course **V5A2 – The Habiro Ring of a Number Field** taught by Prof. Peter Scholze at the Universität Bonn in the Winter 2024/25 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Recordings of the lecture are availible at the following link:

`archive.mpim-bonn.mpg.de/id/eprint/5132/`

Contents

1. Lecture 1 – 11th October 2024

This course will roughly center around forthcoming joint work of the instructor with Stavros Garoufalidis, Campbell Wheeler, and Don Zagier [GS+24] and will largely focus on the theme of "Ramanujan-style" q-series.

One motivation for the this circle of ideas is to elucidate the connection between the q of "Ramanujan-style" q-series to the q that arises in the study of p-adic Hodge theory through p-adic period rings like Fontaine's $\mathbb{A}_{\text{inf}}$ as in [Sch17]. The correspondence between rings and affine schemes hints at the possibilility for these q-series to be studied as functions on an appropriately defined space. However, it remains unclear what such a topological space should be.

Consider the case of the q-Pochhammer symbol, one of the most elementary q-series.

Definition 1.1 (q-Pochhammer)**.** The q-Pochhammer symbol $(t;q)_n$ is given by the product

$$\prod_{i=0}^{n-1}(1-tq^i).$$

Consider the symbol $(q;q)_\infty$ given by the product $\prod_{n=1}^{\infty}(1-q^n)$ which are naturally elements of the power series ring $\mathbb{Z}[[q]]$. In fact, the product under consideration can be rescaled such that $q(q;q)_\infty^{24}$ admits an expansion $\sum_{n\geq 0}\tau(n)q^n$ where $\tau(n)$ is Ramanujan's τ-function. Now contemplating the expansion

$$q\prod_{n=0}^{\infty}(1-q^n)^{24}$$

one can deduce that the function above is a full level of weight 12 – converging on $q\in\mathbb{C}$ with $|q|<1$, on $q=\exp(2\pi iz)$ with z having positive imaginary part, satisfy a periodicity condition, and an automorphy condition under $\mathrm{SL}_2(\mathbb{Z})$ transformations on the complex upper half-plane $\mathbb{H}$. Scaling appropriately, one realizes $\Delta(q)=q(q;q)_\infty^{24}$ as a full level modular form of weight 12.

The discussion above suggests that the quotient of $\mathbb{H}$ by the action of $\mathrm{SL}_2(\mathbb{Z})$ by linear fractional transformations will play a key role in the theory, with the additional automorphy data kept track of via the data of a line bundle ω on the quotient, with $\Delta(q)$ above being a global section of the line bundle $\omega^{\otimes 12}$ on the quotient space. Indeed, while the quotient space $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ begins life as a Riemann surface, it is in fact not only a complex-analytic curve, but also defined over $\mathbb{Z}$ as a modular curve. In this light, $\Delta(q)$ is not only a complex-valued section of the line bundle $\omega^{\otimes 12}$ over the complex numbers, but a section of the same line bundle defined over the integers, and as such carries arithmetic information.

Over the course of the semester, we will unravel a similar story by consdering q-series and their corresponding (non-)modularity properties in conjunction with their associated spaces and line bundles for which these q-series naturally arise as sections.

These considerations begin with a discussion of Nahm sums, the definition of which we now recall.

Definition 1.2 (Nahm Sum)**.** Let A be a positive-definite $N \times N$ matrix over $\mathbb{Z}$. The Nahm sum $f_A(q)$ is given by

$$f_A(q) = \sum_{n \in \mathbb{Z}^N_{\geq 0}} \frac{q^{\frac{1}{2}n^T An}}{\prod_{i=1}^N (q;q)_{n_i}} \in \mathbb{Z}[[q^{1/2}]].$$

Remark 1.3. This definition can be generalized, taking a positive-definite $N \times N$ matrix over $\mathbb{Q}$, and yielding an element of $\mathbb{Z}[[q^{1/2d}]]$ for d large.

Remark 1.4. In fact, the $N = 1$ case considering Nahm sums of the form

$$f_a(q) = \sum_{n \geq 0} \frac{q^{\frac{1}{2}an^2}}{(q;q)_n}$$

already contains the essential features of the theory, and we will restrict to this case when the technical difficulties of $n \geq 2$ obfuscate the clarity of exposition.

Using simple estimates, one can show that these Nahm sums are convergent for $q \in \mathbb{C}$ with $|q| < 1$.

Even in the simplest case of $N = 1$ and $a = 1$, we can get surprising "modular-type" behavior and yielding a modular function.

Proposition 1.5. Let $f_1(q) = \sum_{n \geq 0} \frac{q^{\frac{1}{2}n^2}}{(q;q)_n}$. Then $f_1(q) = q^{-1/48}(-q^{1/2};q)_\infty$.

Proof. We will show for $f_1(t,q) = \sum_{n \geq 0} \frac{q^{\frac{1}{2}n^2}}{(q;q)_n} t^n \in \mathbb{Z}[[q^{1/2}, t]]$ that $f_1(t,q) = (-q^{1/2}t;q)_\infty$ and recover the desired result for $t = 1$.

We show that the q-difference equation $f_1(t,q) - f_1(qt,q) = q^{1/2}t \cdot f_1(qt,q)$ is satisfied. Computing, we have

$$\begin{aligned} f_1(t,q) - f_1(qt,q) &= \left(\sum_{n \geq 0} \frac{q^{\frac{1}{2}n^2}}{(q;q)_n} t^n \right) + \left(\sum_{n \geq 0} \frac{q^{\frac{1}{2}n^2+1}}{(q;q)_n} t^n \right) \\ &= \sum_{n \geq 0} \frac{q^{\frac{1}{2}n^2}}{(q;q)_n} t^n (1 - q^n) \\ &= \sum_{n \geq 0} \frac{q^{\frac{1}{2}(n-1)^2 + (n-1) + \frac{1}{2}}}{(q;q)_{n-1}} t^{n-1} t \\ &= q^{1/2} t \cdot f_1(qt,q) \end{aligned}$$

which is the unique solution to the q-difference equation congruent to 1 modulo t by inspecting the $n = 0$ term of $f_1(t,q)$. As such, $f_1(t,q) = (1 + q^{1/2}t) f_1(qt,q)$ and iterating this procedure, we express $f_1(t,q)$ as the desired infinite product. ∎

The techniques of t-deformations and finding solutions to q-difference equations – which can be thought of as a q-analogue of the classical derivative – will play a crucial role in the development of the theory. This setup will allow us to prove results by "analytic continuation in the t-variables." More generally, we can consider Nahm sums of the form $f_A(t_1, \dots, t_N, q)$ specialized to $t_1 = \cdots = t_N = 1$ arising from multivariate t-deformations with $f_A(t_1, \dots, t_N, q)$ being the unique solution 1 modulo $t_1, \dots, t_N$ to a system of q-difference equations.

In the situation above, as $q \to 1$ we can think of this as analogous to taking a deriviative. However, unlike the theory of linear differential equations that can be studied through the rich and well-studied geometry of D-modules, these q-difference equations do not carry the local data necessary to set up a geometric theory in a parallel fashion to that of D-modules. Yet, the fact that these q-difference equations arise in the study of arithmetic geometry, in particular q-de Rahm cohomology which are q-deformed analogues of de Rahm cohomology, reaffirms the possibility of a geometric theory.

Returning to our discussion of modularity properties, in the case of $a = 2$ is well known with

$$f_2(q) = \sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} = (q;q^5)_\infty^{-1}(q^4;q^5)_\infty^{-1} \tag{1.1}$$

being one of the Rogers-Ramanujan identities [OEISa], and is a modular function up to multiplication by $q^{-1/60}$.

It is perhaps surprising, then, that for no other integers, and no rational numbers other than $a = \frac{1}{2}$, is the corresponding Nahm sum a modular function. In fact, it will be possible to provide a systematic explanation as to why it is only these special values of a that result in a modular function, and a reason for the reason for the repeated power of five in both q-Pochhammer symbol factors of (1.1).

By considering systems of t-deformed Nahm equations of the form

$$1 - z_i = t_i z_1^{A_{i1}} z_2^{A_{i2}} \dots z_N^{A_{iN}}$$

where A_{ij} is the i,jth entry of the $N \times N$ matrix A that on specialization to $t_1 = \cdots = t_N = 1$, has a unique solution $z_i \in (0,1)$ for all i. This produces a number field

$$\mathbb{K} = \mathbb{Q}(z_1, \dots, z_N). \tag{1.2}$$

Furthermore, there is a class $\xi \in K_3(\mathbb{K})$ the third algebraic K-group of the number field $\mathbb{K}$ which is known to be isomorphic to the Bloch group $B(\mathbb{K})$, with the isomorphism taking $\xi \in K_3(\mathbb{K})$ to $\sum_{i=1}^N [z_i] \in B(\mathbb{K})$.

Nahm conjectured that the modularity of $f_A(q)$ was completely determined by the behavior of $\xi \in K_3(\mathbb{K})$, in particular that $f_A(q)$ is modular if and only if ξ is torsion. There is a fairly clear understanding of how modularity implies torsion on the class in the K-group through results of Calegari-Garoufalidis-Zagier [CGZ21], but the converse is much less understood.

The result of Calegari-Garoufalidis-Zagier can already be observed in the modularity of the Rogers-Ramanujan identity previously discussed: the Nahm equation

is given by $1-z=z^2$ with root the golden ratio $\frac{1+\sqrt{5}}{2}$ giving rise to the number field $\mathbb{Q}(\sqrt{5})$. This number field is totally real, and as such the entire K-group $K_3(\mathbb{K})$ is torsion as shown by Borel. The square root of five appearing here also explains the structure of the product in (1.1) by the splitting behavior of primes in this field.

For the proof of this direction of Nahm's conjecture, Calegari-Garoufalidis-Zagier study the asymptotics of Nahm sums at roots of unity. In particular, while Nahm sums blow up at roots of unity, one can still consider the behavior of power series as one approaches roots of unity radially

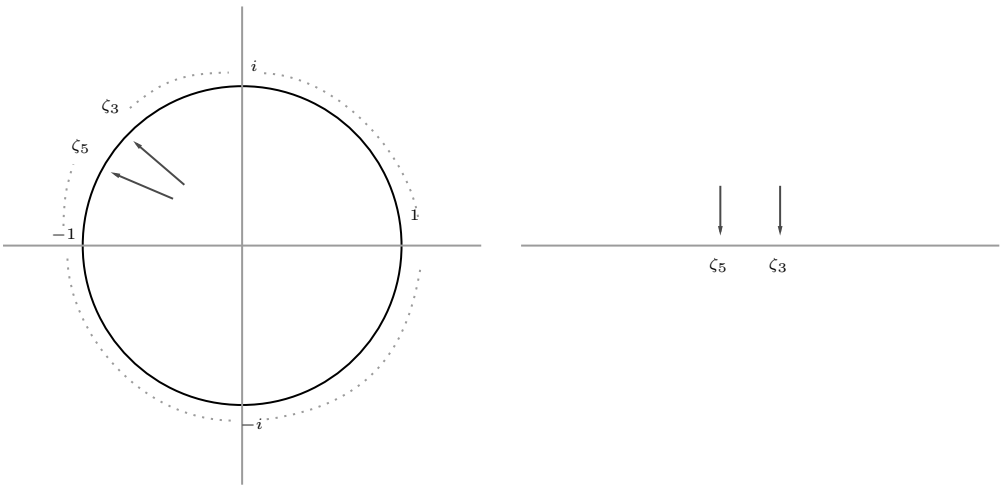


FIGURE 1. Approaching roots of unity radially in the disc and from above in the upper-half plane.

We can then translate this to the upper-half plane model, where we then consider asymptotics as we approach the points with imaginary part zero. If the function were modular, we know that the modular gorup acts on the boundary of $\mathbb{H}$ with finitely many orbits, so understanding these asymptotics becomes much simpler, since we expect conjugate asymptotics to be closely related, even if they are not equal on the nose. In fact, Garoufalidis and Zagier are able to describe the asymptotic behavior of these functions based using lifts of ξ by dilogarithms which turns out to be the complex regulator of the K-group.

However, understanding these asymptotics on approach to roots of unity is closely related to a construction made by Habiro in the 2000s [Hab04].

Definition 1.6 (Habiro Ring)**.** The Habiro ring $\mathcal{H}$ is

$$\lim_{n,m\geq 1} \mathbb{Z}[q]/(1-q^m)^n = \lim_{n} \mathbb{Z}[q]/(q;q)_n$$

the completion of $\mathbb{Z}[q]$ at all roots of unity.

Elements of the Habiro ring are of the form $(f_m)_m \in \mathbb{Z}[\zeta_m][[q-\zeta_m]]$ but for m, pm the rings are the same up to p-adic completion. That is, $f_m = f_{pm} \in \mathbb{Z}[\zeta_{pm}][[q-$

$\zeta_m]] \cong \mathbb{Z}_p[\zeta_{pm}][[q-\zeta_{pm}]]$. As such, these power series agree in characteristic p but no other characteristic.

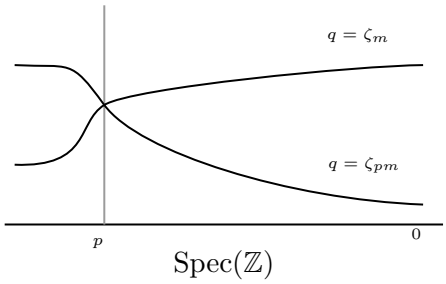


FIGURE 2. The power series f_m and f_{pm} agree in characteristic p.

The power series arising from Nahm sums, however, live in the number field $\mathbb{K}$, necessitating an extended definition of the Habiro ring that works over more than just the integers. Moreover, one can compute $f_A(q)f_A(q^{-1})$ which lie in $\mathcal{O}_{\mathbb{K}}[\frac{1}{\mathrm{disc}(\mathbb{K})}][\zeta_m][[q-\zeta_m]]$ but the gluing doesn't work naïvely, though it can be shown that these agree up to a lift of Frobenius on the p-adic completion $(\mathcal{O}_{\mathbb{K}})^{\wedge}_p$. The prediction of Garoufalidis-Zagier is that there is a natural map from $K_3(\mathbb{K})$ to the Picard groupoid of line bundles on the (Zariski) spectrum of the yet-to-be-defined Habiro ring of a number field $\mathcal{H}_{\mathbb{K}}$ such that the asymptotic expansions at roots of unity of f_A define sections of $\mathcal{H}_{\mathbb{K},\xi}$. We will explore this theorem and its implications over the course of the semester.

2. Lecture 2 – 18th October 2024

Let us revisit the q-Pochhammer function as an example of Nahm sums, and in particular to consider the asymptotic behavior of these rings as q approaches roots of unity. Indeed, the study of such phenomena is precisely the study of the Habiro ring.

We consider the theory of q-calculus and some of its more modern incarnations. The following table summarizes the analogy.

Classical	**q-deformed**	**q-deformed (adapted)**
$\nabla : \mathbb{Z}[t] \to \mathbb{Z}[t]$	$\nabla_q : \mathbb{Z}[q][t] \to \mathbb{Z}[q][t]$	$\nabla'_q : \mathbb{Z}[q][t] \to \mathbb{Z}[q][t]$
$f(t) \mapsto \lim_{h\to 0} \frac{f(t)-f(t+h)}{h}$	$f(t) \mapsto \frac{f(t)-f(qt)}{t-qt}$	$f(t) \mapsto \frac{f(t)-f(qt)}{t}$
$t^n \mapsto nt^{n-1}$	$t^n \mapsto \frac{1-q^n}{1-q} \cdot t^{n-1} = [n]_q t^{n-1}$	$t^n \mapsto (1-q^n)t^{n-1}$

Table 1. Comparison between classical calculus and two variants of q-calculus.

The q-deformed construction recovers the classical case as $q \to 1$, but the adapted q-deformed variant often works better since the lack of $(1-q)$-factors "treats all roots of unity the same," unlike in the classical q-deformed variant which "singles out the first root of unity."

Note that ∇ is coordinate independent as a local operator, but ∇_q is not, and multiplication by q is information that needs to be remembered. However, despite these issues, q-de Rahm cohomology groups turn out to be coordinate independent after $(q-1)$-adic completion, as implied by the theory of prismatic cohomology as developed in [BS22]. In the ∇'_q-variant, however, this coordinate independence does not hold as shown by Wagner in [Wag22] (vis. [Wag24]), but recent work of Meyer-Wagner shows the theory does still hold at some level of generality [MW24].

Let us now consider q-integration, to the end of considering solutions to q-difference equations. Classically, $\nabla f(t) = f(t)$ with initial value $f(0) = 1$ yields the power series $f(t) = \sum_{n\geq 0} \frac{t^n}{n!} = \exp(t)$. In the q-deformed setting, we have the following.

Proposition 2.1. The q-difference equation with $\nabla_q f(t) = f(t)$ with initial value $f(0) = 1$ has solution $f(t) = \sum_{n\geq 0} \frac{t^n}{[n]_q!}$ where $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$.

Proof. This can be verified by a direct computation:

$$\begin{aligned}\nabla_q f(t) &= \sum_{n\geq 0} \frac{[n]_q t^{n-1}}{[n]_q!} \\ &= \sum_{n\geq 0} \frac{t^{n-1}}{[n-1]_q!} = f(t)\end{aligned}$$

which satisfies the initial value condition by inspection. ∎

Similarly in the case of adapted q-deformations, we have the following.

Proposition 2.2. The q-difference equation with $\nabla'_q f(t) = f(t)$ with initial value $f(0) = 1$ has solution $f(t) = \sum_{n\geq 0} \frac{t^n}{(q;q)_n}$.

Proof. Computing as above:

$$\begin{aligned}\nabla'_q f(t) &= \sum_{n\geq 0} \frac{(1-q^n)t^{n-1}}{(q;q)_n} \\ &= \sum_{n\geq 0} \frac{t^{n-1}}{(q;q)_{n-1}}\end{aligned}$$

which once again, by observation, satisfies the initial value condition. ∎

Remark 2.3. We will primarily focus on the adapted variant ∇'_q.

Remark 2.4. The above are examples of Nahm sums of Definition 1.2 for the case $N = 1$ and $A = 0$.

In the adapted variant, $f(t)$ can be described similarly.

Proposition 2.5. The q-difference equation with $\nabla'_q f(t) = f(t)$ with initial value $f(0) = 1$ has solution $f(t) = (t;q)_\infty^{-1}$.

Proof. Given $f(t) = \frac{f(t)-f(qt)}{t}$ we have that $(1-t)f(t) = f(qt)$ and thus

$$f(t) = (1-t)f(qt) \tag{2.1}$$

Applying the same manipulation to $f(qt) = \frac{f(qt)-f(q^2t)}{qt}$ we have $f(qt) = (1-qt)f(q^2t)$ which by induction and substituting into (2.1) we get $f(t) = (t;q)_\infty^{-1}$, yielding the claim. ∎

This gives an expression for $(t;q)_\infty^{-1}$.

Corollary 2.6. There is an equality

$$(t;q)_\infty^{-1} = \sum_{n\geq 0} \frac{t^n}{(q;q)_n}$$

in $\mathbb{Z}[q^{\pm}, \frac{1}{1-q}, \frac{1}{1-q^2}, \dots][[t]]$.

Proof. This is immediate from Propositions 2.2 and 2.5. ∎

We are interested in two phenomena:

- the asymptotics as $q \to 1$ recovering the classical theory, and
- the asymptotics at roots of unity.

One immediately observes that these functions have poles at roots of unity, but their logarithms converge as power series in t with coefficients in $\mathbb{Q}(q)$. We show the logarithm of $(t;q)_\infty^{-1}$ has at worst simple poles at roots of unity.

Proposition 2.7. There is an equality

$$\log(t;q)_\infty^{-1} = \sum_{\ell\geq 1} \frac{1}{\ell(1-q^\ell)} \cdot t^\ell \tag{2.2}$$

in $\mathbb{Q}(q)[[t]]$. As such, $\log(t;q)_\infty^{-1}$ has at worst simple poles at all roots of unity.

Proof. We compute

$$\begin{aligned}
\log(t;q)_\infty^{-1} &= \sum_{n\geq 0} \log(1-q^n t)^{-1} \\
&= \sum_{n\geq 0}\sum_{\ell\geq 1} \frac{q^{n\ell}t^\ell}{\ell} && \log(1-x)^{-1} = \sum_{\ell\geq 1}\frac{x^\ell}{\ell} \\
&= \sum_{\ell\geq 1}\left(\sum_{n\geq 0} q^{n\ell}\right)\frac{t^\ell}{\ell} \\
&= \sum_{\ell\geq 1}\left(\frac{1}{1-q^\ell}\right)\frac{t^\ell}{\ell} && \text{sum of geom. series} \\
&= \sum_{\ell\geq 1}\frac{1}{\ell(1-q^\ell)}t^\ell
\end{aligned}$$

giving the first claim.

For the second claim, observe that the denominator of (2.2) vanishes at order at most 1 at roots of unity, yielding the proposition. ∎

We now consider the behavior at $q=1$, and to that end we consider $\log(t;q)_\infty^{-1}$ as an element of $\frac{1}{q-1}\mathbb{Q}[[q-1,t]]$. To simplify computations, we make the variable change $q=\exp(h)$ and writing our power series in $\frac{1}{h}\mathbb{Q}[[h,t]]$ since $\log(q)=\log(1-(q-1))=h$ with $\log(q)$ in $\mathbb{Q}[[q^{-1}]]$ and understand the asymptotic behavior by writing equations as power series in the variable h. To that end, we recall the following definitions.

Definition 2.8 (Bernoulli Number)**.** Let $n\geq 0$. The nth Bernoulli number B_n is the nth coefficient in the power series expansion

$$-\frac{x}{1-e^x} = \sum_{n\geq 0}\frac{B_n}{n!}x^n \in \mathbb{Q}[[x]].$$

Definition 2.9 (Polylogarithm)**.** Let $n\in\mathbb{Z}$. The nth polylogarithm is the function

$$\mathrm{Li}_n(x) = \sum_{\ell\geq 1}\frac{x^\ell}{\ell^n} \in \mathbb{Q}[[x]].$$

Let us consider some elementary properties of the polylogarithm.

Lemma 2.10. The nth polylogarithm satisfies the differential equation $\nabla\mathrm{Li}_n(t) = \frac{1}{t}\mathrm{Li}_{n-1}(t)$ with initial condition $\mathrm{Li}_n(0)=0$.

Proof. We compute

$$\nabla \mathrm{Li}_n(t) = \sum_{\ell \geq 1} \frac{\ell t^{\ell-1}}{\ell^n} = \sum_{\ell \geq 1} \frac{t^{\ell-1}}{\ell^{n-1}}$$

so multiplying by t we have

$$t \cdot \nabla \mathrm{Li}_n(t) = \sum_{\ell \geq 1} \frac{t^\ell}{\ell^{n-1}} = \mathrm{Li}_{n-1}(t)$$

so $\frac{1}{t}\mathrm{Li}_{n-1}(t) = \mathrm{Li}_n(t)$ with the intial condition holding since $\sum_{\ell \geq 1} \frac{0^\ell}{\ell^n} = 0$. ∎

The some small values of the polylogarithm are given below [Wei02a].

$$\mathrm{Li}_{-2}(t) = \frac{t(t+1)}{(1-t)^3} \qquad \mathrm{Li}_{-1}(t) = \frac{t}{1-t}$$
$$\mathrm{Li}_0(t) = \frac{t}{1-t} \qquad \mathrm{Li}_1(t) = -\log(1-t)$$

TABLE 2. Values of $\mathrm{Li}_n(t)$ for $-2 \leq n \leq 1$.

Lemma 2.11. $\mathrm{Li}_n(t) \in t \cdot \mathbb{Z}[t, \frac{1}{1-t}]$ for $n \leq 0$.

Proof. $\mathrm{Li}_0(t)$ satisfies this by Table 2. We proceed by induction, supposing that $\mathrm{Li}_{-k}(t) \in t \cdot \mathbb{Z}[t, \frac{1}{1-t}]$, we have by Lemma 2.10 that $\mathrm{Li}_{-k-1}(t) = t \cdot \nabla \mathrm{Li}_{-k}(t)$. The induction hypothesis implies $\mathrm{Li}_{-k}(t)$ is a $\mathbb{Z}$-linear combination of elements of the form $\frac{t^a}{(1-t)^b}$ so by the quotient rule, the derivative lies in $\mathbb{Z}[t, \frac{1}{1-t}]$ which suffices by the discussion above. ∎

Elements of in the ring $t \cdot \mathbb{Z}[t, \frac{1}{1-t}]$ behave especially nicely with respect to exponentiation.

Lemma 2.12. Let A be a ring of characteristic 0. If $f(x) \in t \cdot A[t, \frac{1}{1-t}][[x]]$ then $\exp(f)$ admits a power series expansion in $\mathrm{Frac}(A)[t, \frac{1}{1-t}][[x]]$ as $x \to 0$.

Proof. Let us write $f(x) = \sum_{n \geq 0} c_n(t) x^n$ with $c_n(t) \in t \cdot A[t, \frac{1}{1-t}]$ depending on n. We compute

$$\begin{aligned} \exp(f) &= \exp\left(\sum_{n \geq 0} c_n(t) x^n\right) \\ &= \prod_{n \geq 0} \exp(c_n(t) x^n) \\ &= \prod_{n \geq 0} \sum_{k \geq 0} \frac{c_n(t)^k}{k!} x^{nk}. \end{aligned}$$

However, for any fixed N, the coefficient of x^N is a polynomial combination of terms $\frac{c_n(t)^k}{k!}$ where $n \leq N, k \leq N$ of which there are only finitely many, in particular given by some restriction of $\prod_{0\leq n\leq N}\sum_{0\leq k\leq N}\frac{c_n(t)^k}{k!}$ which lies in $\mathrm{Frac}(A)[t,\frac{1}{1-t}]$ since each term does. ∎

With this language in hand, we deduce an asymptotic result about the Pochhammer symbol $(t;q)_\infty$.

Proposition 2.13. The q-Pochhammer symbol $(t;q)_\infty$ satisfies the asymptotic formula

$$(t;q)_\infty \sim \exp\left(\frac{\mathrm{Li}_2(t)}{h}\right)\cdot\sqrt{1-t}\cdot O(h) \tag{2.3}$$

as $q\to 1$ with $O(h)\in\mathbb{Q}[t,\frac{1}{1-t}][[h]]$.

Proof. We compute

$$\begin{aligned}
\frac{t^\ell}{\ell(1-q^\ell)} &= \frac{t^\ell}{\ell(1-e^{h\ell})} && q^\ell=(e^h)^\ell=e^{h\ell}\\
&= \frac{h\ell}{1-e^{h\ell}}\cdot\frac{t^\ell}{h\ell^2}\\
&= -\sum_{k\geq 0}\frac{B_k}{k!}(h\ell)^k\cdot\frac{t^\ell}{h\ell^2} && \frac{h\ell}{1-e^{h\ell}}=-\sum_{k\geq 0}\frac{B_k}{k!}(h\ell)^k\\
&= -\sum_{k\geq 0}\left(\frac{t^\ell}{h\ell^2}\cdot\frac{B_k}{k!}\cdot(h\ell)^k\right)
\end{aligned}$$

so applying this to $\log(t;q)_\infty^{-1}$, we have by Proposition 2.7 that

$$\begin{aligned}
-\log(t;q)_\infty^{-1} &= -\sum_{\ell\geq 1}\frac{t^\ell}{\ell(1-q^\ell)}\\
&= \sum_{\ell\geq 1}\left(\sum_{k\geq 0}\frac{t^\ell}{h\ell^2}\cdot\frac{B_k}{k!}\cdot(h\ell)^k\right) && \text{as above}\\
&= \sum_{k\geq 0}\left(\sum_{\ell\geq 1}\frac{t^\ell}{\ell^{2-k}}\right)\frac{B_k}{k!}\cdot h^{k-1}\\
&= \sum_{k\geq 0}\mathrm{Li}_{2-k}(t)\cdot\frac{B_k}{k!}\cdot h^{k-1} && \mathrm{Li}_{2-k}(t)=\sum_{\ell\geq 1}\frac{t^\ell}{\ell^{2-k}}.
\end{aligned}$$

We write this as

$$\mathrm{Li}_2(t)\cdot B_0\cdot\frac{1}{h}+\mathrm{Li}_1(t)\cdot B_1+\sum_{k\geq 2}\mathrm{Li}_{2-k}(t)\cdot\frac{B_k}{k!}\cdot h^{k-1}.$$

Note here that the third summand is a power series in h with coefficients in $\mathbb{Q}[t, \frac{1}{1-t}]$. Exponentiating, we get, up to constants,

$$\exp\left(\frac{\mathrm{Li}_2(t)}{h}\right) \cdot \sqrt{1-t} \cdot O(h)$$

where the second factor follows from $B_0 = \frac{1}{2}$ and $\exp(-\frac{1}{2}\log(1-t)) = \sqrt{1-t}$, and the third factor from applying Lemma 2.12 to the observation above. ■

Remark 2.14. Something similar to Proposition 2.13 is true for all Nahm sums.

Remark 2.15. It is crucial here to do the expansion in terms of h in order to get a simple result. Doing a power series expansion in other variables will necessitate the use of much more complicated functions.

The proofs we have encountered thus far have largely centered around explicit computation, yielding qualitative descriptions of the expansions. The qualitative features of the higher order terms $a_i(t)$ of (2.3) can in fact be defined recursively by integrating lower order terms. The fact that the integrals of these rational functions remain rational without introducing exotic functions hints at the existence of additional underlying structure to these Nahm sums that may allow qualitative behavior to be deduced without explicit computation.

Returning to the broader discussion at hand, Proposition 2.2 suggests that $(t;q)_\infty^{-1}$ is the q-analogue of the exponential function, and recovering the classical exponential as $q \to 1$, but the behavior we have deduced above is indeed much more complicated. This arises as a consequence of working with ∇_q' in place of ∇_q.

To show the asymptotics at other roots of unity, we will require Bernoulli polynomials.

Definition 2.16 (Bernoulli Polynomial)**.** Let $n \geq 0$. The nth Bernoulli polynomial $B_n(t)$ is the nth coefficient in the power series expansion

$$-\frac{xe^{tx}}{1-e^x} = \sum_{n\geq 0} \frac{B_n(t)}{n!} x^n \in \mathbb{Q}[t][[x]].$$

We state some elementary properties of Bernoulli polynomials.

Lemma 2.17. The Bernoulli polynomials satisfy the following identities:

(i) $B_n(0) = B_n$,
(ii) $B_n(t+1) - B_n(t) = nt^{n-1}$, and
(iii) $B_n(k) = B_k + n \cdot \sum_{i=0}^{k-1} i^{n-1}$ for $k \in \mathbb{N}$.

Proof of (i). This is immediate from the definition. We have $-\frac{xe^{0\cdot x}}{1-e^x} = -\frac{x}{1-e^x}$ recovering Definition 2.8. ■

Proof of (ii). The finite difference formula follows from

$$\begin{aligned}\sum_{n\geq 0}\left(B_n(t+1)-B_n(t)\right)\frac{x^n}{t!} &= \frac{xe^{(t+1)x}-xe^{tx}}{e^x-1}\\ &= \frac{xe^{tx}(e^x-1)}{(e^x-1)}\\ &= xe^{tx}\\ &= \sum_{n\geq 0}\frac{t^n}{n!}x^{n+1} \qquad\qquad e^{tx}=\sum_{n\geq 0}\frac{t^n}{n!}x^n\\ &= \sum_{n\geq 0}(nt^{n-1})\cdot\frac{x^n}{n!}\end{aligned}$$

where the equality is given termwise. ∎

Proof of (iii). Rearranging (ii) we get the recursion $B_n(t+1) = nt^{n-1} + B_n(t)$ so by induction for any natural number k we have

$$B_n(k) = B_n(0) + n\sum_{i=0}^{k-1} i^{n-1}$$

as desired. ∎

We list the first few Bernoulli polynomials [Wei02b].

$$\begin{aligned} B_0(t) &= 1 & B_1(t) &= t-\frac{1}{2}\\ B_2(t) &= t^2-t+\frac{1}{6} & B_3(t) &= t^3-\frac{3}{2}t^2+\frac{1}{2}t\\ B_4(t) &= t^4-2t^3+t^2-\frac{1}{30} & B_5(t) &= t^5-\frac{5}{2}t^4+\frac{5}{3}t^3+\frac{1}{6}t\end{aligned}$$

TABLE 3. Bernoulli polynomials $B_n(t)$ for $0 \leq n \leq 5$.

We now treat the asymptotics at other roots of unity, taking $q = \zeta_m \exp(h)$ where ζ_m is a primitive mth root of unity.

Lemma 2.18. Let ζ_m be a primitive mth root of unity and $q = \zeta_m \exp(h)$. Then

$$\frac{1}{\ell(1-q^\ell)}\cdot t^\ell = -\sum_{n\geq 0}\frac{t^\ell}{\ell^{2-n}}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}\cdot B_n\left(\frac{i}{m}\right)\right)\frac{m^{n-1}}{n!}h^{n-1}. \tag{2.4}$$

Proof. We compute

$$\frac{1}{1-q^\ell} = \frac{1}{1-\zeta_m^\ell e^{h\ell}}$$
$$= \frac{1}{1-(\zeta_m^\ell e^{h\ell})^m}\sum_{i=1}^{m-1}(\zeta_m^\ell e^{h\ell})^i \qquad \frac{1}{1-x} = \frac{1+x+\cdots+x^{m-1}}{1-x^m}$$

so for each summand of (2.2) in Proposition 2.7, we have

$$\begin{aligned}
\frac{t^\ell}{\ell(1-q^\ell)} &= \frac{t^\ell}{\ell}\cdot\frac{1}{1-(\zeta_m^\ell e^{h\ell})^m}\sum_{i=0}^{m-1}(\zeta_m^\ell e^{h\ell})^i \\
&= \sum_{i=0}^{m-1}\frac{\zeta_m^{i\ell}e^{ih\ell}}{\ell(1-\zeta_m^{m\ell}e^{mh\ell})}\cdot t^\ell \\
&= \sum_{i=0}^{m-1}\frac{\zeta_m^{i\ell}e^{ih\ell}}{\ell(1-e^{mh\ell})}\cdot t^\ell && \zeta_m^{m\ell}=1 \\
&= \sum_{i=0}^{m-1}\frac{\zeta_m^{i\ell}e^{\frac{i}{m}x}}{1-e^x}\cdot\frac{t^\ell}{\ell} && x = mh\ell \\
&= \frac{t^\ell}{\ell}\sum_{i=0}^{m-1}\zeta_m^{i\ell}\left(\frac{e^{\frac{i}{m}x}}{1-e^x}\right) \\
&= \frac{t^\ell}{\ell}\sum_{i=0}^{m-1}\zeta_m^{i\ell}\cdot\frac{1}{x}\left(\frac{xe^{\frac{i}{m}x}}{1-e^x}\right) \\
&= \frac{t^\ell}{\ell}\sum_{i=0}^{m-1}\zeta_m^{i\ell}\frac{1}{x}\left(-\sum_{n\geq 0}\frac{B_n(\frac{i}{m})}{n!}x^n\right) && \text{Definition 2.16} \\
&= -\frac{t^\ell}{\ell}\sum_{i=0}^{m-1}\zeta_m^{i\ell}\left(\sum_{n\geq 0}\frac{B_n(\frac{i}{m})}{n!}x^{n-1}\right) \\
&= -\frac{t^\ell}{\ell}\sum_{n\geq 0}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}\cdot B_n\left(\frac{i}{m}\right)\right)\frac{x^{n-1}}{n!} \\
&= -\frac{t^\ell}{\ell}\sum_{n\geq 0}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}\cdot B_n\left(\frac{i}{m}\right)\right)\frac{m^{n-1}h^{n-1}\ell^{n-1}}{n!} && x = mh\ell \\
&= -\sum_{n\geq 0}\frac{t^\ell}{\ell^{2-n}}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}\cdot B_n\left(\frac{i}{m}\right)\right)\frac{m^{n-1}}{n!}h^{n-1}
\end{aligned}$$

giving an expression of the power series in terms of h. ∎

We will require the following statements in the subsequent discussion

Lemma 2.19. The dilogarithm satisfies the identity

$$\frac{1}{m^{n-1}} \cdot \mathrm{Li}_n(t^m) = \sum_{i=0}^{m-1} \mathrm{Li}_n(\zeta_m^i t) \tag{2.5}$$

for $m, n \in \mathbb{N}$.

Proof. We have

$$\begin{aligned}\sum_{i=0}^{m-1} \mathrm{Li}_n(\zeta_m^i t) &= \sum_{i=0}^{m-1} \left(\sum_{\ell \geq 1} \frac{(\zeta_m^i t)^\ell}{\ell^n} \right) \\ &= \sum_{\ell \geq 1} \left(\sum_{i=0}^{m-1} \zeta_m^{i\ell} \right) \frac{t^\ell}{\ell^n}\end{aligned}$$

and now noting

$$\sum_{i=0}^{m-1} \zeta_m^{i\ell} = \begin{cases} m & m | \ell \\ 0 & m \nmid \ell \end{cases}$$

the summands above vanish if ℓ is not a multiple of m so the sum is in fact given by the sum over m-multiples

$$\sum_{\ell \geq 1} \frac{m t^{m\ell}}{(m\ell)^n} = \frac{1}{m^{n-1}} \sum_{\ell \geq 1} \frac{t^{m\ell}}{\ell^n} = \frac{1}{m^{n-1}} \cdot \mathrm{Li}_n(t^m).$$

■

We recover the behavior $q \to \zeta_m$ as $h \to 0$ so applying the expansion of Lemma 2.18 to Proposition 2.7, we get an asymptotic result as above.

Proposition 2.20. The q-Pochhammer symbol $(t;q)_\infty$ satisfies the asymptotic formula

$$(t;q)_\infty \sim \exp\left(\frac{\mathrm{Li}_2(t^m)}{m^2 h} \right) \cdot \frac{\sqrt{1-t^m}}{\prod_{i=0}^{m-1} \cdot (1-\zeta_m t)^{i/m}} \cdot O(h) \tag{2.6}$$

as $q \to \zeta_m$ with ζ_m a primitive mth root of unity and $O(h) \in \mathbb{Q}(\zeta_m)[t, \frac{1}{1-t^m}][[h]]$.

Proof. We compute

$$\begin{aligned}-\log(t;q)_\infty^{-1} &= -\sum_{\ell \geq 1} \frac{1}{\ell(1-q^\ell)} \cdot t^\ell \\ &= \sum_{\ell \geq 1} \left(\sum_{n \geq 0} \frac{t^\ell}{\ell^{2-n}} \left(\sum_{i=0}^{m-1} \zeta_m^{i\ell} \cdot B_n\left(\frac{i}{m}\right) \right) \frac{m^{n-1}}{n!} h^{n-1} \right) \qquad \text{by (2.4)} \\ &= \sum_{n \geq 0} \left(\sum_{\ell \geq 1} \frac{t^\ell}{\ell^{2-n}} \left(\sum_{i=0}^{m-1} \zeta_m^{i\ell} B_n\left(\frac{i}{m}\right) \right) \right) \frac{m^{n-1}}{n!} h^{n-1}\end{aligned}$$

by observation, the terms for $n \geq 2$ are power series in h, and so too is its exponent, so it remains to consider the first two terms of the series given by

$$\begin{aligned}\left(\sum_{\ell\geq 1}\frac{t^\ell}{\ell^{2-n}}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}B_0\left(\frac{i}{m}\right)\right)\right)\frac{1}{mh} &= \left(\sum_{\ell\geq 1}\frac{t^\ell}{\ell^{2-n}}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}\right)\right)\frac{1}{mh} \quad B_0(t)=1\\ &= \left(\sum_{\ell'\geq 1}\frac{mt^{m\ell'}}{(m\ell')^2}\right)\frac{1}{mh}\\ &= \frac{1}{m^2}\left(\sum_{\ell'\geq 1}\frac{t^{m\ell'}}{\ell'^2}\right)\frac{1}{h}\\ &= \frac{1}{m^2h}\mathrm{Li}_2(t^m)\end{aligned}$$

and

$$\begin{aligned}\sum_{\ell\geq 1}\frac{t^\ell}{\ell^{2-n}}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}B_1\left(\frac{i}{m}\right)\right) &= \sum_{\ell\geq 1}\frac{t^\ell}{\ell^{2-n}}\left(\sum_{i=0}^{m-1}\zeta_m^{i\ell}\left(\frac{i}{m}-\frac{1}{2}\right)\right) \qquad \text{Table 2}\\ &= \sum_{i=0}^{m-1}\frac{i}{m}\left(\sum_{\ell\geq 1}\frac{(\zeta_m^i t)^\ell}{\ell}\right)-\frac{1}{2}\sum_{\ell\geq 1}\left(\sum_{i=0}^{\infty}\zeta_m^{i\ell}\right)\\ &= \sum_{i=0}^{m-1}\frac{i}{m}\mathrm{Li}_1(\zeta_m^i t)+\frac{1}{2}\log(1-t^m)\\ &= \frac{1}{2}\log(1-t^m)+\sum_{i=0}^{m-1}\frac{i}{m}\log(1-\zeta_m^i t)\end{aligned}$$

respectively. Exponentiating, we get, up to constants,

$$\exp\left(\frac{\mathrm{Li}_2(t^m)}{m^2h}\right)\cdot\frac{\sqrt{1-t^m}}{\prod_{i=0}^{m-1}\cdot(1-\zeta_m t)^{i/m}}\cdot O(h).$$

■

Qualitatively, this is quite similar to the asymptotic expansion gleaned in (2.3) albeit with a more complicated factor of $O(h)$. In more general settings, the factor of $O(h)$ is the étale regulator maps on algebraic K-theory and becomes increasingly difficult to understand.

3. Lecture 3 – 8th November 2024

Let situate our discussion of the phenomena encountered in Section 2 in a larger context, with an eye towards discussing the observations made here more rigorously over the rest of the semester.

Recall that per Proposition 2.13 that the asymptotics of the q-Pochhammer $(t;q)_\infty$ at $q \to 1$ is given by

$$\exp\left(\frac{\mathrm{Li}_2(t)}{h}\right) \cdot \sqrt{1-t} \cdot O(h)$$

where $O(h) \in \mathbb{Q}[t, \frac{1}{1-t}][[h]]$. Note that the q-Pochhammer naturally admits an expansion in the ring $\mathbb{Q}[[t]]((h))$ of power series in t and Laurent series in h. Thus, dividing by the exponential prefactor, we would expect the ratio

$$\frac{(t;q)_\infty}{\exp\left(\frac{\mathrm{Li}_2(t)}{h}\right)\sqrt{1-t^2}}$$

to lie in $\mathbb{Q}[t, \frac{1}{1-t}][[h]] \subseteq \mathbb{Q}[[t,h]]$ as well. Similarly following Proposition 2.20, we expect the ratio

$$\frac{(t;q)_\infty}{\exp\left(\frac{\mathrm{Li}_2(t^m)}{m^2h}\right)\frac{\sqrt{1-t^m}}{\prod_{i=0}^{m-1}(1-\zeta_m^i t)^{i/m}}}$$

to lie in $\mathbb{Q}(\zeta_m)[t, \frac{1}{1-t^m}][[h]]$ as $q \to \zeta_m$. It is especially surprising that these expansions have good algebraicity properties in the variable t.

Moreover, these power series $O(h)$ appearing above have good p-adic properties, giving arithmetic meaning to the expansions of $(t;q)_\infty$ with the Li_2 term of the abovementioned equations giving the Borel regulator on the third algebraic K-group, and the term $\prod_{i=0}^{m-1}(1-\zeta_m^i t)$ of the latter equation the modulo m étale regulator of the third algebraic K-group.

This peculiar behavior of such power series was first observed by Garoufalidis and Zagier who showed that for a Nahm sum $f_A(q)$ with associated number field $\mathbb{K}$

$$f_A(q) \sim \exp(\mathrm{Li}_2(\xi)) \cdot \frac{\sqrt{\delta}}{\sqrt[m]{\varepsilon_m(\xi)}} \cdot O(h)$$

for some class $\xi \in K_3(\mathbb{K})$ and $O(h) \in \mathbb{K}(\zeta_m)[[h]]$ as $q \to \zeta_m$. Writing $O(h)$ as $g_{A,m}$ and consider

$$g_{A,m} \in \frac{\sqrt{\delta}}{\sqrt[m]{\varepsilon_m(\xi)}} \cdot \mathbb{K}(\zeta_m)[[h]] \cong \frac{\sqrt{\delta}}{\sqrt[m]{\varepsilon_m(\xi)}} \cdot \mathbb{K}(\zeta_m)[[q-\zeta_m]]$$

under the identification $q = \zeta_m \exp(h)$ where $\delta, \varepsilon_m(\xi)$ are transcendental.

Remark 3.1. It is unclear if δ is a function of ξ.

Remark 3.2. These $g_{A,m}$ have huge coefficients – slightly worse than factorial where the degree $p-2$ coefficients have p in the denominator.

One then hopes that to each $g_{A,m}$ and a class $\xi \in K_3(\mathbb{K})$ one can associate a power series

$$g_m(\xi) \in \frac{\sqrt{\delta}}{\sqrt[m]{\varepsilon_m(\xi)}} \cdot \mathbb{K}_p^\wedge(\zeta_m)[[q-\zeta_m]]$$

with $\mathbb{K}_p^\wedge$ the p-adic completion of $\mathbb{K}$ such that the ratio $g_{A,m}/g_m(\xi)$ has p-integral coefficients.

While it is yet uncear how to realize this on the nose, the product

$$h_{A,m}(q) = g_{A,m}(q)g_{A,m}(q^{-1}) \in \mathcal{O}_{\mathbb{K}}\left[\zeta_m, \frac{1}{\Delta}\right][[q-\zeta_m]] \tag{3.1}$$

cancels out the preceding transcendental terms and where Δ is the discriminant of the number field $\mathbb{K}$.

The p-adic properties can be studied in an appropriate generalization of the Habiro ring, but the case of the (ungeneralized) Habiro ring as in Definition 1.6 is already interesting.

Proposition 3.3. Let $\mathcal{H}$ be the Habiro ring. There is an isomorphism

$$\mathcal{H} \cong \left\{(h_m)_{m\geq 1} : \begin{smallmatrix} h_m \in \mathbb{Z}[\zeta_m][[q-\zeta_m]] \text{ such that for all primes } p \\ h_m = h_{pm} \in \mathbb{Z}_p[\zeta_m][[q-\zeta_m]] \cong \mathbb{Z}_p[\zeta_{pm}][[q-\zeta_{pm}]] \end{smallmatrix}\right\}.$$

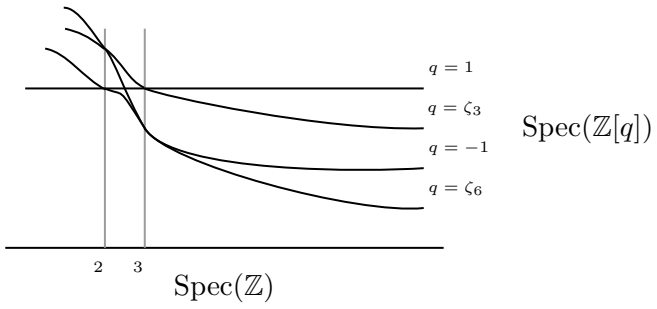


FIGURE 3. p-adic behavior of Habiro ring elements.

Corollary 3.4. There is an injection $\mathcal{H} \hookrightarrow \mathbb{Z}[[q-1]]$.

Corollary 3.5. The map $\mathcal{H} \to \prod_{m\geq 1} \mathbb{Z}[\zeta_m]$ by $h \mapsto (h(\zeta_m))_m$ is injective.

This behavior suggests that the Habiro ring is much more structured than a ring of functions that admits asymptotic expansions at all roots of unity, and is in this sense quite rigid. This is observed, for example, in the p-adic "coherence" behavior of the series appearing in the description of Proposition 3.3.

Unfortunately, this naïve definition of the Habiro ring cannot be nicely generalized. Garoufalidis and Zagier observed that for $h_{A,m} \in \mathcal{O}_{\mathbb{K}}\left[\zeta_m, \frac{1}{\Delta}\right][[q-\zeta_m]]$, it is not the case that $h_{A,m}$ agrees with $h_{A,pm}$ as elements of $\mathcal{O}_{\mathbb{K}_p^\wedge}\left[\zeta_m, \frac{1}{\Delta}\right][[q-\zeta_m]]$, though it does agree up a sign depending on the residue class of the prime p modulo 3 in a special case, suggesting that compatibility could indeed be defined. This allows us to give the eponymous definition.

Definition 3.6 (Habiro Ring of a Number Field)**.** Let $\mathbb{K}$ be a number field with discriminant Δ. The Habiro ring of $\mathbb{K}$ is given by

$$\mathcal{H}_{\mathcal{O}_{\mathbb{K}}[\frac{1}{\Delta}]} = \left\{ (h_m)_{m\geq 1} : \begin{smallmatrix} h_m \in \mathcal{O}_{\mathbb{K}}[\zeta_m, \frac{1}{\Delta}][[q-\zeta_m]] \\ \forall p \nmid \Delta, h_m = \varphi_p(h_{pm}) \in (\mathcal{O}_{\mathbb{K}})_p^{\wedge}[\zeta_m, \frac{1}{\Delta}][[q-\zeta_m]] \end{smallmatrix} \right\}$$

where $(\mathcal{O}_{\mathbb{K}})_p^{\wedge}$ is the p-adic completion of $\mathcal{O}_{\mathbb{K}}$ and φ_p the Frobenius lift on $(\mathcal{O}_{\mathbb{K}})_p^{\wedge}$.

Remark 3.7. $(\mathcal{O}_{\mathbb{K}})_p^{\wedge}$ is a finite étale $\mathbb{Z}_p$-algebra, and the Frobenius lifts uniquely to an endomorphism on the p-adic completion. Étaleness of the $\mathbb{Z}_p$-algebra explains uniqueness of the Frobenius.

While this definition is the appropriate generealization of the Habiro ring, it is yet unclear how explicit elements of this ring can be constructed. In fact, there is no map in general $\mathcal{O}_{\mathbb{K}} \to \mathcal{H}_{\mathcal{O}_{\mathbb{K}}[\frac{1}{\Delta}]}$ – the constant map to power series does not satisfy the identity on the Frobenius lift.

Remark 3.8. In special cases, such as $\mathbb{K}/\mathbb{Q}$ being an Abelian extension, an ad-hoc construction for a map can be made.

This behavior is analogous to the nice behavior of the "symmetrization" of (3.1), and we thus expect that for each $\xi \in K_3(\mathbb{K})$ there is a unique (up to unique isomorphism) locally free rank 1 module or line bundle $L(\xi)$ over $\mathcal{H}_{\mathcal{O}_{\mathbb{K}}[\frac{1}{\Delta}]}$ (resp. $\mathrm{Spec}(\mathcal{H}_{\mathcal{O}_{\mathbb{K}}[\frac{1}{\Delta}]})$) which naturally contains $(g_{A,m})_m$, the power series of asymptotic expansions at roots of unity, as an element (resp. section). In fact, if two matrices A_1, A_2 define the same number field $\mathbb{K}$ and $\xi \in K_3(\mathbb{K})$ as in the procedure of (1.2), then the ratios of the asymptotic expansions at roots of unity

$$\left(\frac{g_{A_1,m}}{g_{A_2,m}}\right) \in \mathrm{Frac}\left(\mathcal{H}_{\mathcal{O}_{\mathbb{K}}[\frac{1}{\Delta}]}\right)$$

which produces a natural map $K_3(\mathbb{K}) \to \mathsf{Pic}(\mathcal{H}_{\mathcal{O}_{\mathbb{K}}[\frac{1}{\Delta}]})$ to the Picard groupoid of line bundles/invertible modules over the Habiro ring of the number field.

As we will soon see, all the expectations above hold, and will be made precise in what follows.

4. Lecture 4 – 15th November 2024

We consider some properties of the dilogarithm function following [Zag07].

Recall polylogarithms from Definition 2.9. We can in fact alternatively define polylogarithms using the relation of Lemma 2.10.

Lemma 4.1. The dilogarithm $\mathrm{Li}_2(t)$ satisfies the integral equation

$$\mathrm{Li}_2(t) = -\int_0^t \log(1-z)dz \tag{4.1}$$

on the slit plane $\mathbb{C} \setminus [1, \infty)$.

Proof. The claim follows from Lemma 2.10 and $\mathrm{Li}_1(t) = -\log(1-t)$ of Table 2, noting that the logarithm is defined on the slit plane. ∎

Special values of the dilogarithm are given as follows [Zag07, §1].

$\mathrm{Li}_2(0) = 0$	$\mathrm{Li}_2(1) = \frac{\pi^2}{6}$	$\mathrm{Li}_2(-1) = -\frac{\pi^2}{12}$
$\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log(2)^2$	$\mathrm{Li}_2\left(\frac{1-\sqrt{5}}{2}\right) = \frac{\pi^2}{15} + \frac{1}{2}\log\left(\frac{1+\sqrt{5}}{2}\right)^2$	

Table 4. Special values of the dilogarithm.

The dilogarithm has especially interesting behavior at roots of unity where it can be expressed in terms of the Dedekind Zeta function and exhibit close connections to L-functions [Zag07, §5]. Dilogarithms also satisfy a number of functional equations:

$$\mathrm{Li}_2(1/t) = -\mathrm{Li}_2(t) - \frac{\pi^2}{6} - \frac{1}{2}\log(-t)^2 \tag{4.2}$$

$$\mathrm{Li}_2(1-z) = -\mathrm{Li}_2(t) + \frac{\pi^2}{6} - \log(t)\log(1-t) \tag{4.3}$$

Table 5. Functional equations for the dilogarithm.

The phenomena above, where products of logarithms appear, are quite common in the setting of dilogarithm functional equations. The dilogarithm can be analytically continued in the following way.

Theorem 4.2. The function

$$\mathrm{Li}_2(e^h) + h \cdot \mathrm{Li}_1(e^h) \tag{4.4}$$

is well deined $\mathbb{C} \setminus (2\pi i\mathbb{Z}) \to \mathbb{C}/(2\pi i)^2\mathbb{Z}$.

Proof. Computing, we get

$$\begin{aligned} \frac{d}{dh}\left(\mathrm{Li}_2(e^h) + h \cdot \mathrm{Li}_1(e^h)\right) &= -e^h \cdot \frac{1}{e^h}\mathrm{Li}_1(e^h) + \mathrm{Li}_1(e^h) + h\mathrm{Li}_0(e^h) \\ &= \frac{he^h}{1-e^h} \qquad \mathrm{Li}_0(t) = \frac{t}{1-t} \end{aligned}$$

which is a well-defined meromorphic function on $\mathbb{C}$ with simple poles at integer multiples of $2\pi i$ and residues multiples of $2\pi i$ so the map descends to the quotient. ■

One can also consider an analogue of the dilogarithm is the Bloch-Wigner dilogarithm.

Definition 4.3 (Bloch-Wigner Dilogarithm)**.** The Bloch-Wigner dilogarithm is the function

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log(|z|).$$

Remark 4.4. The Bloch-Wigner dilogarithm is a well-defined continuous function on $\mathbb{C} \cup \{\infty\}$.

The advantage of working with the Bloch-Wigner dilogarithm is that one no longer needs to consider the products of logarithms in functional equations that arose above. We state some functional equations below.

$$D(z) = D\left(1-\frac{1}{z}\right) = D\left(\frac{1}{1-z}\right) = -D\left(\frac{1}{z}\right) = -D(1-z) = -D\left(\frac{-z}{1-z}\right) \tag{4.5}$$

$$D(x) + D(y) + D\left(\frac{1-x}{1-xy}\right) + D(1-xy) + D\left(\frac{1-y}{1-xy}\right) = 0 \tag{4.6}$$

TABLE 6. Functional equations for the Bloch-Wigner dilogarithm.

Let us return to our discussion of asymptotics of Nahm sums as in Section 2. We focus on the case $A = a$ for $a \in \mathbb{Z}$ producing Nahm sums of the form

$$f_a(t,q) = \sum_{n\geq 0} \frac{q^{\frac{1}{2}an^2}}{(q;q)_n} t^n.$$

The q-difference equation this satisfies can be easily deduced.

Proposition 4.5. Let $a \in \mathbb{Z}$. The t-deformed Nahm sum $f_a(t,q) = \sum_{n\geq 0} \frac{q^{\frac{1}{2}an^2}}{(q;q)_n} t^n$ satisfies the t-deformed q-difference equation

$$f_a(t,q) - f_a(qt,q) = tq^{a/2} f_a(q^a t, q). \tag{4.7}$$

Proof. We compute

$$\begin{aligned} f_a(t,q) - f_a(qt,q) &= \sum_{n\geq 0} \frac{q^{\frac{1}{2}an^2}}{(q;q)_n} t^n - \sum_{n\geq 0} \frac{q^{\frac{1}{2}an^2}}{(q;q)_n} q^n t^n \\ &= \sum_{n\geq 0} \left(\frac{q^{\frac{1}{2}an^2}(1-q^n)}{(q;q)_n} \right) t^n \\ &= \sum_{n\geq 0} \frac{q^{\frac{1}{2}a(n+1)^2}}{(q;q)_n} t^{n+1} \qquad \text{reindexing} \\ &= \sum_{n\geq 0} \frac{q^{\frac{1}{2}an^2+an+\frac{a}{2}}}{(q;q)_n} t^{n+1} \\ &= tq^{a/2} \left(\sum_{n\geq 0} \frac{q^{\frac{1}{2}an^2}}{(q;q)_n} q^{an} t^n \right) \\ &= tq^{a/2} f_a(q^a t, q) \end{aligned}$$

as desired. ∎

We can modify the Nahm sum to rid ourselves of the $q^{a/2}$ factor in (4.7).

The exposition that follows is drawn from the erratum discussed at the begining of lecture 5. The reader is encouraged to consult [GZ21] for full details.

Corollary 4.6. Let

$$f_a^{\mathrm{mod}}(t,q) = f_a(q^{-a/2}t, q) = \sum_{n\geq 0} (-1)^{an} \frac{q^{\frac{1}{2}an^2 - \frac{1}{2}an}}{(q;q)_n} t^n$$

Then the modified Nahm sum satisfies the t-deformed q-difference equation

$$f_a^{\mathrm{mod}}(t,q) - f_a^{\mathrm{mod}}(qt,q) = (-1)^a t \cdot f_a^{\mathrm{mod}}(q^a t, q). \tag{4.8}$$

Proof. This follows from a similar computation as in Proposition 4.5. ∎

Remark 4.7. $f_a(t,q) \in \mathbb{Z}[[t, \sqrt{q}]]$ but $f_a^{\mathrm{mod}}(t,q) \in \mathbb{Z}[[t,q]]$.

We will henceforth work with these modified Nahm sums. As previously discussed, we would expect the asymptotic expansion of such a Nahm sum to be expressed as a product of an exponential of a dilogarithm, a square root, and a power series in h as $q \to 1$ and $q = \exp(h)$, mirroring the discussion of Proposition 2.20.

We use the ansatz

$$f_a(t,q) = \exp\left(\frac{V(t)}{h}\right) g_a(t,q) \tag{4.9}$$

$$f_a(qt,q) = \exp\left(\frac{V(qt)}{h}\right) g_a(qt,q) \tag{4.10}$$

in what follows, with $V(t) \in \mathbb{Q}[[t]]$ satisfying $V(0) = 0$. Equations (4.9) and (4.10) takes are related by the formula we now describe.

Proposition 4.8. The equation $V(t)$ of the Nahm equation ansatz satisfies the logarithmic differential equation

$$V(e^h t) = V(t) + h(\partial^{\log} V)(t) + \frac{h^2}{2}((\partial^{\log})^2 V)(t) \tag{4.11}$$

with $\partial^{\log} V(t) = t \cdot V'(t)$.

Remark 4.9. (4.11) of Proposition 4.8 should be thought of as a multiplicative analogue of the Taylor expansion.

Using this, we can compute the ratio of the leading factors of Equations (4.9) and (4.10).

Proposition 4.10. The Nahm sum $f_a(t,q)$ satisfies the asymptotic formula

$$f_a(t,q) \sim \exp\left(\frac{V(t)}{h}\right) \cdot O(h) \tag{4.12}$$

as $q \to 1$ and $O(h) \in \mathbb{Q}[[t,h]]$.

Corollary 4.11. Let $V(t)$ be as in the Nahm equation ansatz. The ratios of the exponential factors satisfy

$$\exp\left(\frac{V(qt)}{h}\right) / \exp\left(\frac{V(t)}{h}\right) \sim \exp((\partial^{\log} V)(t))(1 + O(h))$$

for $q = e^h$.

Proof. We have by Proposition 4.8

$$\begin{aligned}\exp\left(\frac{V(qt)}{h}\right) &= \exp\left(\frac{V(t)}{h} + (\partial^{\log} V)(t) + \frac{h}{2}((\partial^{\log})^2 V)(t)\right) \\ &= \exp\left(\frac{V(t)}{h}\right) \exp\Big((\partial^{\log} V)(t)\Big) \exp\left(\frac{h}{2}((\partial^{\log})^2 V)(t)\right)\end{aligned}$$

where the first term of the product above cancels in the ratio and the final term of the product expands to a power series in h with constant coefficient 1. ∎

Denoting $Z(t) = \exp((\partial^{\log} V)(t))$ and $\widetilde{Z}(t,q) = \exp\big((\partial^{\log} V)(t)\big) \exp\big(\frac{h}{2}((\partial^{\log})^2 V)(t)\big)$, we divide (4.8) by the exponential prefactor $\exp\left(\frac{V(t)}{h}\right)$ and the ansatz Equations (4.9) and (4.10) to observe

$$g_a(t,q) - \widetilde{Z}(t,q) g_a(qt,q) = (-1)^a \cdot t \cdot \prod_{i=0}^{a-1} \widetilde{Z}(q^i t, q) \cdot g_a(q^a, t)$$

which we seek to show lies in $\mathbb{Q}[[t, q-1]]$.

Specializing at $q = 1$ (ie. $h = 0$) produces

$$1 - Z(t) = (-1)^a t \cdot Z(t)^a \tag{4.13}$$

We now explicitly describe $V(t)$. As suggested by the preceding discussion, it suffices to solve the differential equation $V'(t) = \frac{1}{t}\log(Z(t))$.

Proposition 4.12. Let $V(t)$ be as in the Nahm equation ansatz and $Z(t) = \exp((\partial^{\log} V)(t))$. $V(t)$ satisfies the equation

$$V(t) = -\mathrm{Li}_2(1 - Z(t)) - \frac{a}{2}\log(Z(t))^2 \tag{4.14}$$

with $V(0) = 0$.

Proof. We compute

$$\begin{aligned} V'(t) &= -\frac{Z'(t)}{1 - Z(t)}\log(Z(t)) - a \cdot \log(Z(t)) \cdot \frac{Z'(t)}{Z(t)} \\ &= \log(Z(t)) \cdot \frac{(Z(t) + a(1 - Z(t)))\, Z'(t)}{(1 - Z(t))Z(t)} \end{aligned}$$

and differentiating (4.13) we get $-Z'(t) = (-1)^a Z(t)^a + (-1)^a at \cdot Z(t)^{a-1} Z'(t)$. Now note $(-1)^a at \cdot Z(t)^{a-1} = \frac{1-Z(t)}{Z'(t)}$ so $-Z'(t) = \frac{Z'(t)}{Z(t)}\left(Z(t) + a(1 - Z(t))\right) = -\frac{1-Z(t)}{t}$. Substituting this into the equation for $V'(t)$ we get the desired result. ∎

We can also discuss asymptotics of Nahm sums at roots of unity ζ_m. In parallel to Proposition 2.20, we have the following result for Nahm sums.

Theorem 4.13. The Nahm sum $f_a(t, q)$ satisfies the asymptotic formula

$$f_a(t, q) \sim \exp\left(\frac{V(t^m)}{m^2 h}\right) \cdot O(h) \tag{4.15}$$

as $q \to \zeta_m$ with ζ_m a primitive mth root of unity and $O(h) \in \mathbb{Q}(\zeta_m)[[t, h]]$.

Proof Outline. We use the ansatz

$$f_a(t, q) = \exp\left(\frac{V(t^m)}{m^2 h}\right) g_a(t, h) \tag{4.16}$$

where we want to show that $g_a(t, h) \in \mathbb{Q}(\zeta_m)[[t, h]]$ where *a priori*, $g_a(t, q) \in \mathbb{Q}(\zeta_m)((h))[[t]]$.

By Proposition 4.12, we have that $t \cdot V'(t) = \log(Z(t))$ so we can write

$$\widetilde{Z}(t^m, h) = \frac{\exp\left(\frac{V(q^m t^m)}{m^2 h}\right)}{\exp\left(\frac{V(t^m)}{m^2 h}\right)} = Z(t^m)^{1/m} \cdot (1 + O(h)).$$

and thus

$$g_a(t, h) - \widetilde{Z}(t^m, h) g_a(\zeta_m e^h t, h) = (-1)^a t \cdot \prod_{j=0}^{a-1} \widetilde{Z}(e^{\pi i h} t^m, h) \cdot g_a(\zeta_m^a e^{ah} t, h). \tag{4.17}$$

shwoing specialization at $h = 0$ is well-defined. As such, we can write $g_a(t, 0)$ as

$$\sum_{j=0}^{a-1} t^j h_j(t^m)$$

and h_j a function in t. As such, by (4.17) we have

$$\sum_{j=0}^{a-1} t^j h_j(t^m) - Z(t^m)^{1/m} \sum_{j=0}^{a-1} \zeta_m^j t^j h_j(t^m) = (-1)^a t \cdot Z(t^m)^{a/m} \cdot \sum_{j=0}^{a-1} \zeta_m^{aj} t^j h_j(t^m)$$

and we separate the sum for each j by taking residue classes of exponents of t where we have

$$\begin{aligned} h_j(t^m)(1 - \zeta_m^j Z(t^m)^{1/m}) &= (-1)^a \zeta_m^{a(j-1)} Z(t^m)^{a/m} \cdot h_{j-1}(t^m) && j > 0 \\ h_0(t^m)(1 - Z(t^m)^{1/m}) &= (-1)^a \zeta_m^{a(m-1)} t^m Z(t^m)^{a/m} h_{m-1}(t^m) && j = 0 \end{aligned}$$

where

$$\prod_{j=0}^{m-1} (1 - \zeta_m^j Z(t^m)^{1/m}) = (-1)^{am} \prod_{j=0}^{m-1} \zeta_m^{aj} t^m Z(t^m)^a$$

holds as it simplifies to

$$1 - Z(t^m) = (-1)^a t^m \cdot Z(t^m)^a$$

by factorization results for cyclotomic polynomials. ■

5. Lecture 5 – 22nd November 2024

We continue our discussion of q-series and in particular a property of the modified Nahm sum considered in Corollary 4.6.

The following definition is due to Konsevich-Soibelman [KS11].

Definition 5.1 (Admissable Series)**.** A series $f \in \mathbb{Z}((q))[[t]]$ such that $f \equiv 1 \pmod{(t)}$ is admissable if it can be written as

$$f = \prod_{n\geq 1}\prod_{i\in\mathbb{Z}}(q^i t^n;q)_\infty^{a_{n,i}}$$

such that for each n only finitely many $a_{n,i}$ are nonzero.

Remark 5.2. These $a_{n,i}$'s are precisely Donaldson-Thomas invariants that arise in Gromov-Witten theory and enumerative geometry.

Admissable series force an algebraicity condition on the q, allowing f to be written as an element of $\mathbb{Z}[q][[t]]$. Up to a condition on the residue of the series f mod (t), series in $\mathbb{Z}((q))[[t]]$ admit such an expansion.

Proposition 5.3. Let $f \in \mathbb{Z}((q))[[t]]$. If $f \equiv 1 \pmod{(t)}$ then f admits a unique expansion as a series of the form

$$f = \prod_{n\geq 1}\prod_{i\in\mathbb{Z}}(q^i t^n;q)_\infty^{a_{n,i}}.$$

Proof Outline. This can be solved for truncated polynomials so for $f \equiv 1 \pmod{(t)}$, it suffices to consider $f \equiv 1 \pmod{(t^m)}$ and solve inductively to give an expression algebraic in t. ∎

This result is in fact much more general and it can be shown that the modified Nahm sum

$$f_a(t,q) = \sum_{n\geq 0}(-1)^{an}\frac{q^{\frac{1}{2}an^2-\frac{1}{2}an}}{(q;q)_n}t^n$$

as previously defined is admissable. The original proof is highly involved, and we will instead offer a simpler exposition of the same result. Recall from Proposition 2.7, we have

$$(q^i t;q)_\infty = \exp\left(-\sum_{\ell\geq 1}\frac{1}{\ell}\cdot\frac{q^{i\ell}t^{n\ell}}{1-q^\ell}\right). \tag{5.1}$$

Furthermore, $\mathbb{Z}((q))[[t]]$ is a λ-ring as we now define.

Definition 5.4 (λ-Ring)**.** A λ-ring A is a ring equipped with set maps $\lambda^k : A \to A$ for $0 \leq k \leq \infty$ such that

(i) $\lambda^0(a) = 1$ for all $a \in A$.
(ii) $\lambda^1(a) = a$ for all $a \in A$.
(iii) $\lambda^n(a+b) = \sum_{i+j=n}\lambda^i(a)\lambda^i(b)$ for all $a, b \in A$.

Such a ring admits Adams operations $\psi_n : \mathbb{Z}((q))[[t]] \to \mathbb{Z}((q))[[t]]$ by $t \mapsto t^n, q \mapsto q^n$. The Adams operations allow us to rewrite (5.1) as

$$\exp\left(-\sum_{\ell\geq 1}\psi_\ell\left(\frac{q^i t^n}{1-q}\right)\right). \tag{5.2}$$

We introduce the notion of the plethystic exponential.

Definition 5.5 (Plethystic Exponential)**.** Let A be a λ-ring and $\sum_{\ell\geq 1}\frac{a_\ell}{\ell}$ a convergent series in A. The plethystic exponential is of the series is given by

$$\exp\left(\sum_{\ell\geq 1}\frac{1}{\ell}\psi_\ell(a_\ell)\right).$$

Now taking

$$\phi(t,q) = -\sum_{n\geq 1}\sum_{i\in\mathbb{Z}} a_{n,i}q^i t^n \in \mathbb{Z}((q))[[t]]$$

we have

$$\prod_{n\geq 1}\prod_{i\in\mathbb{Z}}(q^i t;q)_\infty = \exp\left(\sum_{\ell\geq 1}\frac{1}{\ell}\psi_\ell\left(\frac{\phi(t,q)}{1-q}\right)\right).$$

We can define the plethystic logarithm as the inverse of the plethystic exponential and observe $\frac{\phi(t,q)}{1-q}$ is the plethystic logarithm of the modified Nahm sum $f_a(t,q)$. We then seek to show that this plethystic logarithm $\frac{\phi(t,q)}{1-q} \in \frac{1}{1-q}\mathbb{Z}[q^\pm][[t]]$ which has a single simple pole at $q=1$. Indeed, this suffices as the behavior at other roots of unity are determined by the Adams operations.

Observe the plethystic exponential gives an isomorphism $t\mathbb{Q}[q^\pm][[t]] \to 1+t\mathbb{Q}[q^\pm][[t]]$. It thus suffices to show that the plethystic logarithm of f_a is a function that is a sum $\frac{\phi_0(t)}{1-q}$ with an element of $\mathbb{Q}[q^\pm][[t]]$.

Now using the ansatz

$$f_a(t,q) = \exp\left(\sum_{\ell\geq 1}\frac{1}{\ell}\frac{\phi_0(t^\ell)}{1-q^\ell}\right) g_a(t,q) \tag{5.3}$$

we show that there exists a chioce of function $\phi_0(t)$ satisfying the ansatz above would imply the factor $g_a(t,q) \in 1+t\mathbb{Q}[q^\pm][[t]]$ and from which the result of showing $g_a(t,q)$ lying in $1+t\mathbb{Q}[t^\pm][[t]]$ would follow by application of the Plethystic exponential.

But a choice of $\phi_0 \in t\cdot\mathbb{Q}[[t]]$ can be made such that $\sum_{\ell\geq 1}\frac{\phi_0(t^\ell)}{\ell^2} = -V(t)$ whose plethystic exponential has leading term asymptotics agreeing with $f_a(t,q)$ at all roots of unity.

This is gives the desired result as stated below.

Theorem 5.6 (Kontsevich-Soibelman, Efimov)**.** The q-series

$$f_a(t,q) = \sum_{n\geq 0} (-1)^{an} \frac{q^{\frac{1}{2}an^2-\frac{1}{2}an}}{(q;q)_n} t^n$$

is admissable.

Let us unfurl some of the consequences of Theorem 5.6.

To understand the $\phi_0(t)$ function better, we use compute its logarithmic derivative so in conjunction with Proposition 4.8 we have

$$\sum_{\ell\geq 1} \frac{(\partial^{\log}\phi_0)(t^\ell)}{\ell} = -\log Z(t)$$

with $Z(t)$ is the logarithmic derivative of $V(t)$. Note that $Z(t) \in 1 + t\mathbb{Z}[[t]]$ so $(\partial^{\log}\phi_0)(t) = -\sum_{n\geq 1} c_n t^n$ for $Z(t) = \prod_{n\geq 1}(1-t^n)^{c_n}$ for $c_n \in \mathbb{Z}$. This shows $n|c_n$.

Example 5.7. For $a = 0$, $f_1(t,q) = (t;q)_\infty^{-1}$ so $Z(t) = 1 - t$ and $\phi_0(t) = \pm t$ which agrees with $V(t)$ being the dilogarithm (up to a sign).

Example 5.8. In the case $a = 2$ which was discussed in Section 1, we recover $Z(t)$ as an alternating sum of the Catalan numbers.

6. Lecture 6 – 29th November 2024

We make some recollections from algebraic number theory, largely following Milne's texts.

Definition 6.1 (Dedekind Zeta Function)**.** Let $\mathbb{K}$ be a number field. The Dedekind zeta function is given by

$$\zeta_{\mathbb{K}}(s) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_{\mathbb{K}}} \frac{1}{\mathrm{Nm}(\mathfrak{a})^s} = \prod_{\mathfrak{p}\subseteq\mathcal{O}_{\mathbb{K}}} (1-\mathrm{Nm}(\mathfrak{p})^{-s})^{-1}.$$

As in the case of the Riemann zeta function, the Dedekind zeta function can be analytically continued to a meromorphic function on $\mathbb{C}$ with simple pole at $s=1$. The class number formula gives the behavior of the function at the simple pole $s=1$. We state an adapted variant. Recall the definition of the Γ-function.

Definition 6.2 (Γ-Function)**.** The Γ-function is given by

$$\Gamma(s) = \int_0^\infty \frac{t^s e^t}{t} dt.$$

We use the following normalizations

$$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)$$
$$\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s).$$

and the L-series

$$L_{\mathbb{K}}(s) = \Gamma_{\mathbb{R}}(s)^{r_1}\Gamma_{\mathbb{C}}(s)^{r_2}\zeta_{\mathbb{K}}(s) \tag{6.1}$$

in what follows. The class number formula states the following.

Theorem 6.3 (Class Number Formula)**.** Let $\mathbb{K}$ be a number field. Then

$$\lim_{s\to 1}(s-1)L_{\mathbb{K}}(s) = \frac{2^{r_1+r_2}h_{\mathbb{K}}}{w_{\mathbb{K}}\sqrt{\Delta}}\cdot \mathrm{Reg}_{\mathbb{K}}$$

where $w_{\mathbb{K}}$ is the number of roots of unity, $h_{\mathbb{K}}$ the class number, Δ the discriminant of $\mathbb{K}$, and r_1, r_2 the real and complex places of $\mathbb{K}$, respectively.

The regulator $\mathrm{Reg}_{\mathbb{K}}$ captures the transcendental part of the special value. See [M-CFT, §V.2] for further discussion. Additionally, by the Dirichlet unit theorem, we have the following:

Theorem 6.4 (Dirichlet Unit)**.** Let $\mathbb{K}$ be a number field. Then $\mathrm{rank}(\mathcal{O}_{\mathbb{K}}^\times) = r_1 + r_2 - 1$ where r_1, r_2 are the real and complex places of $\mathbb{K}$, respectively.

In fact, we can say more. There is a map $\mathcal{O}_{\mathbb{K}}^\times \to \prod_{\nu|\infty}\mathbb{R}$ $\alpha \mapsto (\log(|\alpha|_\nu))_{\nu|\infty}$ which in fact lands in the kernel of the map $\prod_{\nu|\infty}\mathbb{R}\to\mathbb{R}$ by summing the entries – in particular, a map $\mathcal{O}_{\mathbb{K}}^\times \to \ker\left(\prod_{\nu|\infty}\mathbb{R}\to\mathbb{R}\right)$. But the image of the map is a lattice $\Lambda \subseteq \prod_{\nu|\infty}\mathbb{R} = \mathbb{R}^{r_1+r_2-1}$ and the regulator is defined as the quotient of the kernel by this lattice Λ.

Definition 6.5 (Regulator)**.** Let $\mathbb{K}$ be a number field. The regulator of $\mathbb{K}$ is the volume

$$\mathrm{Vol}\left(\ker\left(\prod_{\nu|\infty}\mathbb{R}\to\mathbb{R}\right)/\Lambda\right)$$

where Λ is the image of $\mathcal{O}_{\mathbb{K}}^{\times}$ in the kernel $\ker\left(\prod_{\nu|\infty}\mathbb{R}\to\mathbb{R}\right)$ of the sum map $\prod_{\nu|\infty}\mathbb{R}\to\mathbb{R}$.

Remark 6.6. In particular, the regulator is the determinant of a $(r_1+r_2-1)\times(r_1+r_2-1)$-matrix with entries of the form $\log(|\alpha|_\nu)$ for $\alpha\in\mathcal{O}_{\mathbb{K}}^{\times}$.

Compare the discussion in [M-ANT, §5].

Example 6.7. In the case of a real quadratic field, there is a unique fundamental unit, and the regulator is just the logarithm of the absolute value of the fundamental unit.

Remark 6.8. Conjecturally, it is expected for all $n\geq 2$ that $L_{\mathbb{K}}(n)$ is asymptotically the determinant of a matrix whose entries are renormalized nth polylogarithms evaluated at entries of $\mathbb{K}$.

The instructor did not officially define K-theory at this point. Those without familiarity with ∞-categories should be able to proceed taking K-theory as an algebraic invariant of rings. A more classical discussion can be found in [Wei13, Ch. IV].

The subsequent discussion will utilize the language of algebraic K-theory, which is an algebraic invariant of the stable ∞-category of perfect complexes over a fixed base ring.

Definition 6.9 (K-Theory of a Ring)**.** Let R be a commutative ring and $\mathsf{Perf}(R)$ its stable ∞-category of perfect complexes. The K-theory anima $K(R)$ is the ∞-categorical group completion of $\mathsf{Perf}(R)$.

This process produces a commutative group object in Ani.

Remark 6.10. See [Stacks, Tag 0656] for a (1-categorical) discussion of perfect complexes, and the notes of Hebestreit-Wagner [HW21] and Hilman-McCandless [HM24] for further discussion on algebraic K-theory from the ∞-categorical perspective.

The K-groups of the ring R are then defined to be the homotopy groups of these animae.

Definition 6.11 (K-Groups of a Ring)**.** Let R be a commutative ring. The ith K-group is the group $K_i(R)=\pi_i K(R)$.

One can, with little danger think of these animae, or ∞-groupoids/Kan complexes, as spaces.

Remark 6.12. In the case $i=0$, this recovers the Grothendieck group of vector bundles on $\mathrm{Spec}(R)$, the group completion of isomorphism classes of vector bundles modulo the scissor relation.

A result of Quillen-Borel shows the K-groups of rings of integers of number fields are finitely generated.

Theorem 6.13 (Quillen-Borel, Soulé; [FG05, I.5, Thm. 6, 7])**.** Let $\mathbb{K}$ be a number field. Then

$$\operatorname{rank}(K_{2n-1}(\mathcal{O}_{\mathbb{K}})) = \begin{cases} r_2 & n \equiv 0 \pmod 2 \\ r_1 + r_2 & n \equiv 1 \pmod 2, n > 1 \end{cases}$$

and $K_i(\mathcal{O}_{\mathbb{K}})$ is torsion when i is even and positive.

We focus on the case $n = 2$ considering K_3 of the number field. More generally, we have the following result of Borel relating the value of the L-series of (6.1) can be related to certain Borel regulators.

Definition 6.14 (Borel Regulator)**.** Let $\mathbb{K}$ be a number field. The n-th Borel regulator $\operatorname{Reg}_{\mathbb{K}}(n)$ is the volume of the quotient $(P_n/\Lambda)/(P_n/\Lambda')$ where P_n is the space of primitives in $H_n(\mathrm{SL}(R), \mathbb{R})$, Λ the image of $K_n(\mathcal{O}_{\mathbb{K}})$ in $H_n(\mathrm{SL}(R), \mathbb{R})$, and Λ' the image of the symmetric space.

Remark 6.15. See [Wei13, §IV.1.18.1] for an expanded discussion.

These are related to values of the L-series as follows:

Theorem 6.16 (Borel; [Wei13, §IV.1.18.1])**.** Let $\mathbb{K}$ be a number field. The L-series satsifies the asymptotic formula

$$L_{\mathbb{K}}(n) \sim \operatorname{Reg}_{\mathbb{K}}(n). \tag{6.2}$$

To prove the conjecture of Remark 6.8, it in fact suffices to understand these K-groups rationally – that is, up to base change to $\mathbb{Q}$. Questions of this type remain actively investigated, but we will restrict our attention to $n = 2$ as previously indicated.

Definition 6.17 (Second Pre-Bloch Group)**.** Let F be a field. The second pre-Bloch group $\wp_2(F)$ is the quotient of $\mathbb{Q}[F^\times \setminus \{1\}]$ by the $\mathbb{Q}$-vector subspace generated by the five term relations of the Bloch-Wigner dilogarithm.

Definition 6.18 (Second Bloch Group)**.** The second Bloch group $B_2(F)$ is the kernel

$$\ker\left(\wp_2(F) \to \bigwedge^2 (F^\times \otimes_{\mathbb{Z}} \mathbb{Q})\right)$$

of the map $[x] \mapsto [x] \wedge [1 - x]$.

The relation of Definition 6.18 is closely related to Milnor K-theory, in particular related in the following way, which can be deduced from [FG05, §1.5, Thm. 8].

Theorem 6.19 (Bloch)**.** Let $\mathbb{K}$ be a number field. Then there are isomorphisms

$$K_3(\mathbb{K}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong B(\mathbb{K}).$$

Moreover, the Borel regulator map is given by the Bloch-Wigner dilogarithm.

Remark 6.20. Conjecturally for $n \geq 2$, we would expect that we could inductively define $\wp_n(F)$ to be the quotient of $\mathbb{Q}[F^\times \setminus \{1\}]$ by the $\mathbb{Q}$-vector subspace generated by functional equations of the nth polylogarithm and relate the weight n Goncharov-Zagier complex

$$\wp_n(F) \longrightarrow \wp_{n-1}(F) \otimes_{\mathbb{Z}} F^\times \longrightarrow \wp_{n-2}(F) \otimes_{\mathbb{Z}} \bigwedge^2 F^\times \longrightarrow \dots$$

(6.3)

$$\dots \longrightarrow \wp_2(F) \otimes_{\mathbb{Z}} \bigwedge^{n-2} F^\times \longrightarrow \bigwedge^n (F^\times \otimes_{\mathbb{Z}} \mathbb{Q})$$

to the K-theory of the number field.

7. Lecture 7 – 6th December 2024

Let us consider the construction of algebraic K-theory as in Definition 6.9.

For a commutative ring R, we can consider the anima $\mathsf{Proj}(R)$ consisting of finitely generated projective R-modules which lies in $\mathsf{CMon}(\mathsf{Ani})$, the commutative monoid objects in the category of anima, under the direct sum operation. However, within $\mathsf{Proj}(R)$ we can consider the full subcategory spanned by the free R-modules $\mathsf{Free}(R)$ which can be obtained as $\coprod_{n\geq 0} */\mathrm{GL}_n(R)$ acting by automorphisms on free modules of rank n for each n. The inclusion induces a map $\mathsf{Free}(R)^{\infty-\mathsf{Grp}} \to \mathsf{Proj}(R)^{\infty-\mathsf{Grp}}$ where $(-)^{\infty-\mathsf{Grp}}$ denotes group completion as an anima. In fact, this suffices to compute algebraic K-theory in strictly positive degrees.

Proposition 7.1. Let R be a ring. The map of anima $\mathsf{Free}(R)^{\infty-\mathsf{Grp}} \to \mathsf{Proj}(R)^{\infty-\mathsf{Grp}}$ is an isomorphism on homotopy groups for $i \geq 1$.

This provides a way to compute the K-theory of R via the homotopy, and in fact homology, of $*/\mathrm{GL}_\infty(R)$. And while *a priori* this seems like an exceptionally daunting task, homological stability shows that it suffices to compute $H_i(*/\mathrm{GL}_n(R))$ since the map

$$H_i(*/\mathrm{GL}_n) \longrightarrow H_i(*/\mathrm{GL}_{n+1})$$

is an isomorphism for n sufficiently large with respect to i. For $R = \mathcal{O}_{\mathbb{K}}$ for $\mathbb{K}$ a number field, $K_i(\mathcal{O}_{\mathbb{K}})$ is a finitely generated Abelian group by a result of Borel Theorem 6.13. Moreover, in these cases, the phenomena are well-studied as the (co)homology of arithmetic groups, which are closely connected to automorphic forms.

Example 7.2. $H_i(*/\mathrm{SL}_2(\mathbb{Z})) \cong H_i(\mathbb{H}^{\pm}/\mathrm{SL}_2(\mathbb{Z}))$ where $\mathbb{H}^{\pm}$ is the union of the upper and lower half plane and the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^{\pm}$ by Möbius transformations. In particular, the desired homology group of $*/\mathrm{SL}_n(\mathbb{Z})$ can be computed as the homology of an arithmetic locally symmetric space $\mathbb{H}^{\pm}/\mathrm{SL}_2(\mathbb{Z})$ which on passage to the Borel-Serre compactification is a manifold with corners.

More generally for an arithmetic group Γ, the homology of the quotient $*/\Gamma$ can be computed as the homology of the Borel-Serre compactification of an associated arithmetic locally symmetric space which is a manifold with corners.

We now make some recollections from condensed mathematics to the end of defining condensed K-theory which rseults from considering $\coprod_{n\geq 0} */\mathrm{GL}_n(\mathbb{C})$ as a condensed anima. The upshot of this approach is that it is able to preserve topological information such as local compactness, instead of treating $\mathrm{GL}_n(\mathbb{C})$ as a mere abstract group.

Definition 7.3 (Condensed Set)**.** A condensed set is a sheaf of sets on the site of profinite sets and coverings given by finite families of jointly surjective maps.

Naïvely, one would expect that we could define the condensed K-theory anima $K^{\mathsf{cond}}(\mathbb{C})$ of $\mathbb{C}$ as the sheafification of the presheaf

$$S \mapsto K(\mathrm{Cont}(S, \mathbb{C}))$$

where passing to homotopy groups recovers K-theory in some fixed degree as a condensed Abelian group.

Remark 7.4. As is typical in the condensed setting $S = *$ recovers $K(\mathbb{C})$ which is the ordinary K-theory anima.

One notices, however that the desired homotopy groups $\mathbb{C}/(2\pi i)^n\mathbb{Z}$ are quite similar to products of $\mathbb{R}/\mathbb{Z}$ which are locally compact Abelian groups that satisfy Pontryagin duality.

Definition 7.5 (Continuous K-Theory Anima)**.** The continuous K-theory anima $K^{\mathsf{cont}}(\mathbb{C})$ is the Pontryagin bidual of the sheafification of the presheaf

$$S \mapsto K(\mathrm{Cont}(S, \mathbb{C})).$$

The homotopy groups of the continuous K-theory anima were computed by Clausen to be the following.

Theorem 7.6 (Clausen)**.** The homotopy groups of the continuous K-theory anima are given as follows:

$$\pi_i K^{\mathsf{cont}}(\mathbb{C}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{C}/(2\pi i)^n\mathbb{Z} & i = 2n-1 \\ 0 & i \equiv 0 \pmod 2, i > 1. \end{cases}$$

The proof of Theorem 7.6 boils down to the computation of condensed cohomology of $*/\mathrm{GL}_n(\mathbb{C})$ with $\mathbb{R}$ or $\mathbb{Z}$-coefficients. With $\mathbb{Z}$-coefficients, the computation of the integral homology of $\coprod_{n\geq 1} */|\mathrm{GL}_n(\mathbb{C})|$ gives topological K-theory ku whose homotopy groups are $\mathbb{Z}$ in all even degrees and zero otherwise. In the case of $\mathbb{R}$-coefficients, this is a Lie algebra computation.

Remark 7.7. Conjecturally, the K-theory of the liquid and gaseous complex numbers are given by

$$K_i(\mathbb{C}_{\mathsf{Liq}}) = K_i(\mathbb{C}_{\mathsf{Gas}}) = \begin{cases} \mathbb{Z} & i \leq 0, i \equiv 0 \pmod 2 \\ 0 & i = 0 \text{ or } i > 0 \text{ and } i \equiv 0 \pmod 2 \\ \mathbb{C}/(2\pi i)^n\mathbb{Z} & i > 0 \text{ and } \equiv 1 \pmod 2. \end{cases}$$

Moreover, it is expected that this recovers periodic K-theory KU.

One can also provide a condensed account of Beilinson's construction of K-theory by $\mathbb{C}_{\mathsf{Liq}}$ or $\mathbb{C}_{\mathsf{Gas}}$ the liquid and gaseous complex numbers, respectively.

Recall from the preceding discussion that for a field F, $K_0(F) \cong \mathbb{Z}$ classifing the dimension of finite dimensional vector spaces and $K_1(F) = \mathrm{GL}_\infty(F)^{\mathsf{ab}} \cong F^\times$. In the case of $F = \mathbb{C}$, we can define $\mathbb{C}^\times \cong K_1(\mathbb{C}) = K_1^{\mathsf{cont}}(\mathbb{C}) \cong \mathbb{C}/(2\pi i)\mathbb{Z}$ where the maps are given by the exponential and the logarithm. More generally, there is a map

$$K_{2n-1}(\mathcal{O}_\mathbb{K}) \to \left(\prod_{\tau:\mathbb{K}\to\mathbb{C}} \mathbb{C}/(2\pi i)^n\mathbb{Z}\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})} \tag{7.1}$$

which is Galois equivariant under the action of complex conjugation on both $\mathbb{C}$ and $(2\pi i)^n\mathbb{Z}$ agree, and hence lands in the Galois invariant part of the product. In particular, for a complex place the value of one embedding is already determined by the other, but for a real place there is only one term in the product which already lies in the Galois invariant part of the quotient. Now, on taking the real part if n is odd, and the imaginary part if n is even, the map (7.1) above extends to a map

$$K_{2n-1}(\mathcal{O}_{\mathbb{K}}) \to \left(\prod_{\tau:\mathbb{K}\to\mathbb{C}} \mathbb{R}\right)^{\mathrm{Gal}(\mathbb{C}/\mathbb{R})}. \tag{7.2}$$

Depending if n is even or odd, the real places may or may not contribute since real places are invariant under the Galois action. Thus the target of (7.2) is

$$\begin{cases} \mathbb{R}^{r_1+r_2} & n \equiv 1 \pmod 2 \\ \mathbb{R}^{r_2} & n \equiv 0 \pmod 2 \end{cases}$$

Given a number field $\mathbb{K}$, each embedding $\tau : \mathbb{K} \to \mathbb{C}$ gives rise to a regulator map $K_{2n-1}(\mathcal{O}_{\mathbb{K}}) \to K^{\mathsf{cont}}_{2n-1}(\mathbb{C}) \cong \mathbb{C}/(2\pi i)^n\mathbb{Z}$.

8. Lecture 8 – 13th December 2024

We discuss the computation of algebraic K-theory groups of a field F in degrees ≤ 3. In fact, the first nontrivial case is K_3 as the 0th, 1st, and 2nd K-theory groups are given by $\mathbb{Z}$, $F^\times$, and $K_2^{\mathsf{M}}(F)$ the Milnor K-theory of F, respectively.

In the case of interest, computing K_3 of a number field, it suffiecs to understand the homology of $\mathrm{GL}_2(F)$ in degrees at most 3. To do so, we consider the action of $\mathrm{GL}_2(F)$ on $\mathbb{P}^1(F)$ by linear fractional transformations. The computation relies on the following lemma.

Proposition 8.1. Let $(\mathbb{P}^1(F))^n_{\neq}$ be the set of n pairwise distinct points on $\mathbb{P}^1(F)$. There is a functorial exact complex

$$\cdots \to \mathbb{Z}[(\mathbb{P}^1(F))^2_{\neq}]_{\Sigma_2} \to \mathbb{Z}[\mathbb{P}^1(F)] \to \mathbb{Z} \to 0$$

where $(-)_{\Sigma_n}$ denotes the coinvariants of the natural action of the symmetric groups which is an exact complex of $\mathrm{GL}_2(F)$-modules.

The resolution of Proposition 8.1 produces a spectral sequence converging to $H_{i+j}(*/\mathrm{GL}_2(F))$ with E_1-page given as in (A.1). We compute the homology in each case. We do so degree by degree in the complex of Proposition 8.1

Proposition 8.2. The action of $\mathrm{GL}_2(F)$ on $\mathbb{P}^1(F)$ by linear fractional transformations is transitive, and the stabilizer of ∞ is given by the Borel subgroup $\mathcal{B}_2(F)$ of upper triangular matrices. Moreover,

$$H_i(*/\mathcal{B}_2(F)) \otimes \mathbb{Q} \cong H_i(*/(F^\times)^2) \cong \bigwedge^i \left((F^\times)^2 \otimes \mathbb{Q}\right).$$

In the case of $\mathbb{Z}[(\mathbb{P}^1(F))^2_{\neq}]_{\Sigma_2}$, we have the following.

Proposition 8.3. The homology of $H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^2_{\neq}]_{\Sigma_2}\right) \otimes \mathbb{Q}$ is given by

$$H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^2_{\neq}]_{\Sigma_2}\right) \otimes \mathbb{Q} = \begin{cases} 0 & i = 0 \\ F^\times \otimes \mathbb{Q} & i = 1 \\ \bigwedge^2((F^\times)^2 \otimes \mathbb{Q})_{\Sigma_2} & i = 2. \end{cases}$$

And in degrees $\mathbb{Z}[(\mathbb{P}^1(F))^3_{\neq}]_{\Sigma_3}$ and $\mathbb{Z}[(\mathbb{P}^1(F))^4_{\neq}]_{\Sigma_4}$, we have the following.

Proposition 8.4. The action of $\mathrm{GL}_2(F)$ on $(\mathbb{P}^1(F))^3_{\neq}$ factors over the simply transtive action of $\mathrm{PGL}_2(F)$ on $(\mathbb{P}^1(F))^3_{\neq}$. In particular, the homology vanishes.

Proposition 8.5. The 0th homology of $\mathbb{Z}[(\mathbb{P}^1(F))^4_{\neq}]_{\Sigma_4}$ is given by

$$H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^4_{\neq}]_{\Sigma_4}\right) \otimes \mathbb{Q} = \mathbb{Q}[F^\times \setminus \{1\}]_{\Sigma_4}.$$

This produces the necessary data for the E_1-page. For the E_2-page, we further need to understand $H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^5_{\neq}]_{\Sigma_5}\right) \otimes \mathbb{Q}$.

Proposition 8.6. The 0th rational homology of $\mathbb{Z}[(\mathbb{P}^1(F))^5_{\neq}]_{\Sigma_5}$ is given the $\mathbb{Q}$-vector subspace of $\mathbb{Q}[F^\times \setminus \{1\}]$ generated by the five term relations of the Bloch-Wigner dilogarithm.

Substituting the results of Propositions 8.2 to 8.5 into (A.1) we get on the E_1-page (A.2). In conjunction with Proposition 8.6, we get on the E_2-page

$$\begin{array}{cccc} \wp(F) & 0 & 0 & \mathbb{Q} \\ & 0 & 0 & ((F^\times)^2 \otimes \mathbb{Q}) \\ & & 0 & \bigwedge^2((F^\times)^2 \otimes \mathbb{Q}) \\ & & & \bigwedge^3((F^\times)^2 \otimes \mathbb{Q}) \end{array} \tag{8.1}$$

with no differentials. On the E_3-page, the map $\wp(F) \to \bigwedge^2((F^\times)^2 \otimes \mathbb{Q})$ recovers the second Bloch group Definition 6.18 as the kernel and thus Bloch's result Theorem 6.19 for $F = \mathbb{K}$ a number field.

Remark 8.7. For this computation, one can also instead consider the action of the Picard groupoid $\mathsf{Pic}(F)$ on the K-theory anima as in Definition 6.9 and consider the homotopy orbits $K(F)_{h(*/F^\times)} \otimes \mathbb{Q}$ which also recovers the pre-Bloch group as its third homotopy group. This exhibits $K(F)_{h(*/F^\times)}$ as the group completion of

$$\left(\coprod_{n\geq 0} */\mathrm{GL}_n(F)\right)_{h(*/F^\times)}.$$

Recalling Theorem 6.19, and the subsequent discussion we have regulator maps for each complex embedding $\tau : \mathbb{K} \to \mathbb{C}$ a Bloch regulator map $K_3(F) \cong B(F) \to \mathbb{C}/(2\pi i)^2\mathbb{Q}$ which can be extended to a map to $\mathbb{R}$ by taking the imaginary part, with $\sum_i n_i[x_i] \mapsto \sum_i n_i D(\tau(x_i))$, here taking the Bloch-Wigner dilogarithm.

9. Lecture 9 – 20th December 2024

We consider the notion of relative K-theory, a variant of K-theory in which K-theoretic classes are more easily described, and is as computable as K-theory. In this way, relative K-theory can be seen as more flexible than K-theory.

Definition 9.1 (Relative K-Theory)**.** Let R be a ring and M an Abelian group with a map $M \to R^\times$. The relative K-theory

$$K(R/\mathbb{Z}[M]) = K(R) \otimes_{\mathbb{S}[*/M]} \mathbb{S}$$

where $K(R)$ is considered as a spectrum over $\mathbb{S}[*/M]$ by the map $*/M \to */R^\times \to R$.

Remark 9.2. By the universal property of group rings, the existence of a map $M \to R^\times$ is equivalent to the existence of a map from the group ring $\mathbb{Z}[M]$ to R.

Remark 9.3. One can generalize Definition 9.1 to the setting of M a commutative monoid M to R which corresponds to a map $\mathbb{Z}[M] \to R$. One can then define

$$K(R/\mathbb{Z}[M]) = K^{\log}(R, M) \otimes_{\mathbb{S}[*/M^{\mathrm{gp}}]} \mathbb{S}$$

where $K^{\log}$ is logarithmic K-theory.

Remark 9.4. The construction of Definition 9.1 above is equivalent to taking the homotopy orbits $K(R)_{h(*/M)}$ which in the case of F a field produces $K(F)_{h(*/F^\times)} = K(F/\mathbb{Z}[F^\times])$.

The upshot of this construction is that all constructions in K-theory generalize to the setting of relative K-theory. Of importance to us are regulators and polylogarithms. In particular, the dilogarithm is most naturally expressed in the setting of relative K-theory.

Moreover, relative K-theory overcomes the rigidity of K-theory in high weights. For example, it is conjectured that the Bloch group of $\mathbb{C}$ and of $\overline{\mathbb{Q}}$ coincide. On the other hand, classes in relative K-theory is plentiful. As such, the regulators are now highly interesting special functions and not merely numbers and provides a coherent organizing principle for functions like dilogarithms.

These constructions also arise in p-adic geometry. Let R be a smooth algebra over $\mathbb{C}_p$, the p-completion of the algebraic closure of $\mathbb{Q}_p$. We often pick a system of coordinates $T_1, \ldots, T_d \in R$ invertible and consider an étale map $\mathbb{C}_p[T_1^{\pm}, \ldots, T_d^{\pm}] \to R$ and pass to

$$R_\infty = R \otimes_{\mathbb{C}_p[T_1^{\pm}, \ldots, T_d^{\pm}]} \mathbb{C}_p \left[T_1^{\pm \frac{1}{p^\infty}}, \ldots, T_d^{\pm \frac{1}{p^\infty}} \right]$$

which on further passage to an appropriate completion is a perfectoid algebra. This is similar to the data required for relative K-theory since the group generated by $T_1^{\pm}, \ldots, T_d^{\pm}$ naturally maps to the units of R. In this case, the étale p-complete relative K-theory

$$K_{\text{ét}}(R/\mathbb{Z}[T_1^{\pm}, \ldots, T_d^{\pm}])^\wedge_p$$

turns out to be the p-completed étale K-theory $K_{\text{ét}}(R_\infty)$. Morally, what happens here is the p-power roots vanish in the K-theory of the p-completion. See [AKN24]

for a further discussion on this topic, viewing syntomic cohomology as a form of p-adic K-theory.

Going forward, we will take the perspective of polylogarithms being relative K-theory classes and functional equations of polylogarithms are already identities of relative K-theory.

Example 9.5. Let $R = \mathbb{Z}[t^{\pm}, \frac{1}{1-t}]$ and $M = t^{\mathbb{Z}}$. There is a natural map $M \to R^{\times}$. We have

$$K(R/\mathbb{Z}[t^{\pm}]) = K(R) \otimes_{\mathbb{S}[*/t^{\mathbb{Z}}]} \mathbb{S}$$

inducing an exact triangle

$$K(R) \to K(R/\mathbb{Z}[t^{\pm}]) \to K(R/\mathbb{Z}[t^{\pm}])[2]$$

where the map $K(R/\mathbb{Z}[t^{\pm}]) \to K(R/\mathbb{Z}[t^{\pm}])[2]$ can be thought of as a logarithmic t-derivative $\nabla_t^{\log}$ by the exact sequence of $\mathbb{S}[*/t^{\mathbb{Z}}]$-modules

$$\mathbb{S}[*/t^{\mathbb{Z}}] \to \mathbb{S} \to \mathbb{S}[2].$$

In particular, any K-theory class gives rise to a relative K-theory classes and K-theory classes can be recovered from those relative K-theory classes that vanish under the differential.

Now recall the existence of a motivic filtration on K-theory [FS02] and is still a topic of contemporary interest with T. Bouis' recent results in the mixed characteristic case [Bou24]. Running this machinery on relative K-theory, this produces a filtration by relative motivic complexes

$$\mathbb{Z}(n)\left(R/\mathbb{Z}[M]\right)$$

and relative motivic cohomology of a ring relative to a group algebra taking $\mathbb{Z}[M] \to R$ to $\mathbb{Z}(n)(R/\mathbb{Z}[M])$. Note that rationalized K-theory can be recovered from relative motivic cohomology $K(R/\mathbb{Z}[M]) \cong \bigoplus_{n\geq 0} \mathbb{Q}(n)(R/\mathbb{Z}[M])[2n]$.

We have an exact triangle

$$\mathbb{Z}(n)(R) \to \mathbb{Z}(n)\left(R/\mathbb{Z}[t^{\pm}]\right) \to \mathbb{Z}(n-1)\left(R/\mathbb{Z}[t^{\pm}]\right)$$

which in weight ≤ 2 computes K-theory in small degrees. Explicitly, we have $\mathbb{Z}(0)(R) = \mathbb{Z}$ and $\mathbb{Z}(1)(R) = R^{\times}[-1]$. By $\mathbb{A}^1$-invariance for regular rings, we have $K(\mathbb{Z}[t]) \cong K(\mathbb{Z})$ so we have $K(\mathbb{Z}[t^{\pm}, \frac{1}{t-1}]) \cong K(\mathbb{Z}) \oplus K(\mathbb{Z})[1] \oplus K(\mathbb{Z})[1] = K(\mathbb{Z}) \oplus K(\mathbb{Z})[1]^{\oplus 2}$. In the relative setting, the exact triangle allow us to compute relative K-theory

$$\mathbb{Z}(1)(R) \longrightarrow \mathbb{Z}(1)(R/\mathbb{Z}[t^{\pm}]) \longrightarrow \mathbb{Z}(0)(R/\mathbb{Z}[t^{\pm}]) \xrightarrow{1\mapsto t} R^{\times}$$

and where making the appropriate substitutions on rationalization gives

$$0 \longrightarrow \mathbb{Q}(2)(R/\mathbb{Z}[t^{\pm}]) \longrightarrow \mathbb{Q}[-1]$$

so in fact there is an isomorphism $\mathbb{Q}(2)(R/\mathbb{Z}[t^{\pm}]) \to \mathbb{Q}[-1]$ induced by the so-called universal dilogarithm that takes $\mathrm{Li}_2^{\mathrm{univ}}(t)$ to $(1-t)$ where we note that $\mathbb{Q}[-1]$ is generated by $1-t$.

We now want to observe that this so-called universal dilogarithm satisfies the expected functional equations.

Proposition 9.6. There is an equality $\mathrm{Li}_2^{\mathrm{univ}}(t) = -\mathrm{Li}_2^{\mathrm{univ}}(1-t)$ in relative rational motivic cohomology

$$H^1\left(\mathbb{Q}(2)\left(\mathbb{Z}\left[t^{\pm}, \frac{1}{1-t}\right]/\mathbb{Z}[t^{\pm}, (1-t)^{\pm}]\right)\right).$$

Proof. There is a Koszul-like complex computing $\mathbb{Z}(n)(R)$ as the limit of

$$\mathbb{Z}(n)\left(R/\mathbb{Z}[t_1^{\pm}, t_2^{\pm}]\right) \xrightarrow{(\nabla_{t_1}^{\log}, \nabla_{t_2}^{\log})} \mathbb{Z}(n-1)\left(R/\mathbb{Z}[t_1^{\pm}, t_2^{\pm}]\right)^{\oplus 2} \longrightarrow \mathbb{Z}(n-2)\left(R/\mathbb{Z}[t_1^{\pm}, t_2^{\pm}]\right)$$

so noting that $\mathbb{Q}(2)(R) \cong 0$ and

$$\mathbb{Q}(i)\left(\mathbb{Z}\left[t^{\pm}, \frac{1}{1-t}\right]/\mathbb{Z}[t^{\pm}, (1-t)^{\pm}]\right)$$

vanishes for $i \in \{0, 1\}$ we have that

$$\mathbb{Q}(2)\left(\mathbb{Z}\left[t^{\pm}, \frac{1}{1-t}\right]/\mathbb{Z}[t^{\pm}, (1-t)^{\pm}]\right) \cong \mathbb{Q}[-1].$$

So equality follows by considering the composite. ∎

Remark 9.7. In general if the dilogarithm on a ring R the condition of $\sum_i \mathrm{Li}_2(f_i(t))$ being constant implies that $\sum_i f_i(t) \wedge (1 - f_i(t)) = 0$ in $\bigwedge^2 R^{\times}$. This holds for the five-term relation Equation (4.6).

We now consider the Borel (complex) regulator on algebraic K-theory. There is a map

$$K_3\left(\mathbb{Z}\left[t^{\pm}, \frac{1}{1-t}\right]/\mathbb{Z}[t^{\pm}]\right) \to K_3(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}]) \to K_3^{\mathsf{cont}}(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}])$$

where we define $K_3^{\mathsf{cont}}(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}]) = \pi_3\left(K^{\mathsf{cont}}(\mathbb{C}) \otimes_{\mathbb{S}[*/M]} \mathbb{S}\right)$ and which statisfies a functional equation. So for $t \in \mathbb{C} \setminus \{0, 1\}$, its image in $K_3^{\mathsf{cont}}(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}])$ satisfies a functional equation as well.

Rationally, we can consider rationalized relative motivic complexes $\mathbb{Q}(n)^{\mathsf{cont}}(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}])$ where in the case of $\mathbb{Q}(2)^{\mathsf{cont}}(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}])$ we can compute using $\mathbb{Q}(0)^{\mathsf{cont}}(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}]) \cong \mathbb{Q}, \mathbb{Q}(1)^{\mathsf{cont}}(\mathbb{C}/\mathbb{Z}[\mathbb{C}^{\times}]) \cong (\mathbb{C}^{\times}/\mathbb{C}^{\times})[-1] \cong 0$ so the fiber sequence computes $\mathbb{Q}(2)^{\mathsf{cont}}(\mathbb{C})$ as $(\mathbb{C}/(2\pi i)^2\mathbb{Q})[-1]$ and is given by an extension E fitting into the short exact sequence

$$0 \longrightarrow \mathbb{C}/(2\pi i)^2\mathbb{Q} \longrightarrow E \longrightarrow \mathbb{Q} \otimes_{\mathbb{Z}} \bigwedge^2 \mathbb{C}^{\times} \longrightarrow 0$$

where the map $\mathbb{C}\setminus\{0,1\} \to E$ by $t \mapsto (t)\wedge(1-t)$ satisfies the five-term relation. This extends to a map $\mathbb{Q}[\mathbb{C}\setminus\{0,1\}]$ to E which necessarily factors over the pre-Bloch group $\wp_2(\mathbb{C})$ of Definition 6.17 since the map satisfies the five-term relation. This in turn induces a map $B_2(\mathbb{C}) \to \mathbb{C}/(2\pi i)^2\mathbb{Q}$ as expected. More explicitly, E is obtained as the cokernel of $\mathbb{C}\otimes_{\mathbb{Z}}\mathbb{C} \to \mathbb{C}/(2\pi i)^2\mathbb{Q} \oplus \mathbb{C}^{\times}\otimes_{\mathbb{Z}}\mathbb{C}$ by $x\otimes y \mapsto (xy, \exp(x)\otimes y + \exp(y)\otimes x)$.

10. Lecture 10 – 17th January 2025

Observe that we can consider the ring of integers Nahm number field in the relative setting over $\mathbb{Z}[t_1, \ldots, t_N]$ and develop a notion of the relative Habiro ring as follows. Let

$$R = \frac{\mathbb{Z}\left[t_1, \ldots, t_N, z_1, \ldots, z_N, \frac{1}{\Delta}, \frac{1}{z_1(1-z_1)}, \ldots, \frac{1}{z_N(1-z_N)}\right]}{\left(1 - z_i = (-1)^{A_{ii}} t_i z_1^{A_{i1}} \ldots z_N^{A_{iN}}\right)} \tag{10.1}$$

of Equation (1.2) as a ring over $\mathbb{Z}[t_1, \ldots, t_N]$. We seek a relative Habiro ring $\mathcal{H}_{R/\mathbb{Z}[t_1,\ldots,t_N]}$ and an invertible module over this ring such that our Nahm sums are sections of this line bundle.

Recall that for $R = \mathbb{Z}[t, \frac{1}{1-t}]$ as discussed in Section 9, we can define relative motivic cohomology $H^i(\mathbb{Z}(n)(R/\mathbb{Z}[t^{\pm}]))$ where there is a map

$$\nabla_t^{\log} : H^1(\mathbb{Z}(2)(R/\mathbb{Z}[t^{\pm}])) \longrightarrow H^1(\mathbb{Z}(1)(R/\mathbb{Z}[t^{\pm}])) = R[1/t]^{\times}/t^{\mathbb{Z}}$$

taking $-\mathrm{Li}_2^{\mathrm{univ}}(t)$ to $[\frac{1}{1-t}]$. Under "de Rham realization," the diloagarithm class goes to $-\log(1-t)$, but this is precisely the (logarithmic) differential equation that defines the dilogarithm as a power series.

Remark 10.1. More generally, for R as in (10.1), there is a canonical class $V^{\mathrm{univ}} \in H^1\left(\mathbb{Z}(2)(R/\mathbb{Z}[t_1, \ldots, t_N])\right)$ and whose logarithmic derivative under any $\nabla_{t_i}^{\log}$ is the class z_i for all $1 \leq i \leq N$.

More generally, motivic cohomology has multiple realizations such as de Rham, Betti, étale, and even prismatic cohomology. And while the de Rham realization recovers the classical dilogarithm, we can show that the étale realization also gives rise to information appearing in asymptotic expansions of Nahm sums. In particular, it will explain the factor $\prod_{i=0}^{m-1}(1-\zeta_m t)^{i/m}$ at the expansion of Nahm sums at mth roots of unity Theorem 4.13.

For R a ring, motivic cohomology admits Betti and étale realizations

$$H^i(R, \mathbb{Z}(n)) \longrightarrow H^i_{\mathsf{sing}}(\mathrm{Spec}(R)(\mathbb{C}), \mathbb{Z})$$
$$H^i(R, \mathbb{Z}(n)) \longrightarrow H^i_{\mathsf{\acute{e}t}}(\mathrm{Spec}(R[1/m]), \mathbb{Z}/m\mathbb{Z}(n))$$

We can describe this explicitly for first cohomology $i = 1$ in motivic weight $n = 1$.

- (de Rham Realization) Takes $f \in R^{\times}$ to the $\mathbb{Z}$-torsor of choices of the logarithm $\log(f)$.
- (Étale Realization) Recalling that $\mathbb{Z}/m\mathbb{Z}(1) \cong \mu_m$, the realization takes f to the torsor of choices of mth roots of f, which is well-defined up to an mth root of unity.

Analogous constructions can be made for relative motivic cohomology. In relative motivic cohomology, the same constructions yield maps

$$H^i(\mathbb{Z}(n)(R/\mathbb{Z}[t^{\pm}])) \longrightarrow H^i_{\mathsf{sing}}\left(\mathrm{Spec}(R)(\mathbb{C}) \times_{(\mathbb{C}^{\times})} \mathbb{C}, \mathbb{Z}\right)$$
$$H^i(\mathbb{Z}(n)(R/\mathbb{Z}[t^{\pm}])) \longrightarrow H^i_{\mathsf{\acute{e}t}}\left(\mathrm{Spec}(R[1/m, t^{1/m}]), \mathbb{Z}/m\mathbb{Z}(n)\right)$$

but the objects live only over appropriate covers of the spaces $\mathrm{Spec}(R)(\mathbb{C}), \mathrm{Spec}(R[\frac{1}{m}])$ obtained after extracting the logarithm of t, respectively. In the first case, the fibered product is taken over the map above and the $\exp : \mathbb{C} \to (\mathbb{C}^\times)^N$. This reinforces the intuition that relative motivic cohomology should one where the contribution of the motivic cohomology of $\mathbb{Z}[t^{\pm}]$ is ignored, and this is done in singular cohomology by extracting logarithms. Similarly in the étale setting, we consider the space $\mathrm{Spec}(R[1/m, t^{1/m}])$ which plays the role of the unviersal cover in the étale algebraic setting.

We consider the realization of the universal dilogarithm $\mathrm{Li}_2^{\mathrm{univ}}$ in this setting. Recall from Lemma 4.1 and Theorem 4.2 that the function $\mathrm{Li}_2(t) + \log(t)\log(1-t)$ is a well-defined function $\mathbb{C}\backslash(2\pi i)\mathbb{Z} \to \mathbb{C}/(2\pi i)^2\mathbb{Z}$. Note that there is an isomorphism $\mathbb{C}^\times \backslash \{1\} \times_{\mathbb{C}^\times} \mathbb{C}$ with $\mathbb{C}\backslash(2\pi i)\mathbb{Z}$ by taking the logarithm. The $\mathbb{Z}$-torsor obtained by the map on relative motivic cohomology is the given by the $\mathbb{Z}$-torsor of choices of liftings of $\mathbb{C}/(2\pi i)^2\mathbb{Z}$ to $\mathbb{C}$, and moreover is related to the Beilinson-Deligne cohomology. In this framework, it can be seen that the Betti realization of the diloagarithm is a well-defined function on $\mathbb{C} \setminus (2\pi i)\mathbb{Z}$ with at most simple poles at $(2\pi i)\mathbb{Z}$, and residues $\pm(2\pi i)n$ at $2\pi i n$. In particular, the diloagarithm realizes to a $\mathbb{Z}$-local system on $\mathbb{C} \setminus (2\pi i)\mathbb{Z}$ whose mononodromy around $2\pi i n$ is n. Similarly, the étale realization produces a $\mathbb{Z}/m\mathbb{Z}$ local system on $\mathbb{A}^1_{\mathbb{Z}[\frac{1}{m},\zeta_m]} \setminus \mu_m$. But this local system must be compatible with the Betti realization, so the monodromy at each ζ_m^i is congruent to $i \pmod m$ and trivial at zero. But this is sufficient data to determine to determine the torsor, forcing the étale realization to be precisely the product $\prod_{i=0}^{m-1}(1-\zeta_m t)^{i/m}$ alluded to above, and which is the cyclic quantum dilogarithm. This also recovers a construction of Calegari-Garoufalidis-Zagier.

11. Lecture 11 – 24th January 2025

We return to a consideration of the asymptotics of general Nahm sums

$$f_a(t,q) = \sum_{n\geq 0} \frac{q^{\frac{1}{2}an^2}}{(q;q)_n} t^n$$

for $a \in \mathbb{N}$ even. By Theorem 4.13, we have, substituting $\varepsilon = -h$ that

$$f_a(t,q) \sim \exp\left(-\frac{V(t^m)}{m^2\varepsilon}\right) \cdot O(\varepsilon)$$

with $O(h) \in \mathbb{Q}(\zeta_m)[t][[\varepsilon]]$ and

$$V(t) = -\mathrm{Li}_2(1-Z(t)) - \frac{a}{2}\log(Z(t))^2$$

and $Z(t)$ satisfying the ansatz $1 - Z(t) = t \cdot Z(t)^a$. Note that by a even, the $(-1)^a$ factor becones irrelevant.

We want to understand the series $O(h)$, and in particular to show that it lies in the ideal

$$\frac{\sqrt{\delta(t^m)}}{\sqrt[m]{\varepsilon_m}} \cdot R_m[h]$$

and $R_m = R \otimes_{\mathbb{Q}[t]} \mathbb{Q}(\zeta_m)[t]$ along the map $t \mapsto t^m$ for

$$R = \mathbb{Q}\left[t, z, \frac{1}{z(1-z)\delta}\right] / (1 - z = tz^a)$$

$\delta = z + a(1-z)$.

Generally, there are three methods for computing asymptotic expansions:

- (KWB Method) Divide by the exponential prefactor $\exp\left(-\frac{V(t^m)}{m^2 h}\right)$ and get a q-difference equation for the remaining factor $g_a(t,q)$ which can be solved by iterative integration which yields an algebraic function at each step.
- Write
$$f_a(t,q) = \frac{1}{(q;q)_\infty} \sum_{n\in\mathbb{Z}} (q^{n+1};q)_\infty q^{\frac{1}{2}an^2} t^n$$
as $q \to 1$. It can be shown that this quantity is well-approximated by the integral over $\mathbb{R}$ under an appropriate reparametrization.
- (Meinardus' Method) Write $f_a(t,q)$ as the contour integral over S^1
$$f_a(t,q) = \int_{S^1} \left(\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}an^2} (z^a x)^{-n}\right) ((1-z)x, q)_\infty^{-1} \frac{dx}{x}$$
which is convergent as when t is small, z is close to 1 so the second factor is convergent and the Θ-function of the first factor is also convergent.

Remark 11.1. The second and third methods above treat $f_a(t,q)$ as holomorphic functions on the open disc $|q| < 1$ and compute asymptotics of $f_a(t,q)$ as a complex analytic function.

The latter two methods, which rely on complex analysis allow us to leverage the property of the Θ-function which relates the asymptotics as $q \to \zeta_m$ to the asymptotics as $q \to i\infty$.

Proposition 11.2. If $A, B \in \mathbb{C}$ such that $\mathrm{Re}(A) > 0$ then

$$\sum_{n\in\mathbb{Z}} \exp\left(-\frac{1}{2}An^2 - Bn\right) = \sqrt{\frac{2\pi}{A}} \sum_{n\in\mathbb{Z}} \exp\left(\frac{1}{2}\cdot\frac{(B+2\pi in)^2}{A}\right).$$

Proof. Apply Poisson summation which states for functions with all derivatives decaying at ∞ satisfy

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)$$

where $\widehat{f}$ denotes the Fourier transform to $f(x) = \exp\left(-\frac{1}{2}Ax^2 + Bx\right)$.

We have

$$\begin{aligned}
\int_{\mathbb{R}} \exp\left(-\frac{1}{2}Ax^2 + Bx\right)\exp(2\pi ixy)dy &= \int_{\mathbb{R}} \exp\left(-\frac{1}{2}Ax^2 - (B+2\pi iy)x\right)dy \\
&= \int_{\mathbb{R}} \exp\left(-\frac{1}{2}A\left(\frac{B+2\pi iy}{A}\right)^2 + \frac{1}{2}\left(\frac{(B+2\pi iy)^2}{A}\right)\right)dy \\
&= \exp\left(\frac{1}{2}\frac{(B+2\pi iy)^2}{A}\right)\int_{\mathbb{R}} \exp\left(-\frac{1}{2}Ax^2\right)dx \\
&= \exp\left(\frac{1}{2}\frac{(B+2\pi iy)^2}{A}\right)\sqrt{\frac{2\pi}{A}}
\end{aligned}$$

as desired. ∎

Recall that for R as above, we have $R_m = R \otimes_{\mathbb{Q}[t]} \mathbb{Q}(\zeta_m)[t^{1/m}]$ and a class V^{univ} in $H^1(\mathbb{Z}(2)(R/\mathbb{Z}[t]))$ whose étale realization is a class in $H^1_{\text{ét}}(R_m, \mathbb{Z}/m\mathbb{Z}(2))$ where $\mathbb{Z}/m\mathbb{Z}(2) \cong \mu_m$ so $H^1_{\text{ét}}(R_m, \mathbb{Z}/m\mathbb{Z}(2)) \cong R_m^\times/(R_m^\times)^m$, a μ_m-torsor. This produces a $\mathbb{G}_m$-torsor after base-change, that is, a line bundle L_m on $R_m[[h]]$.

Theorem 11.3. The q-difference equation defining $f_a(t,q)$ admits a unique solution 1 modulo t in the module L_m over $R_m[[h]]$.

12. Lecture 12 – 31st January 2025

Recall from Theorem 4.13 and the discussion of the first part of Section 11 that we have

$$f_a(t,q) \sim \exp\left(-\frac{V(t)}{m^2\varepsilon}\right) g_{a,m}(t,q)$$

a convergent series on the open unit disc. We can consider asymptotics as $q \to \zeta_m$ setting $q = \zeta_m \exp(-\varepsilon)$ and $g_{a,m}(t,q) \in \mathbb{Q}(\zeta_m)[t][[\varepsilon]]$. We consider the étale $\mathbb{Z}[t]$-algebra

$$R = \mathbb{Z}\left[t, z, \frac{1}{\delta}\right] / (1 - z = (-1)^a t z^a)$$

with $\delta = z + a(1-z)$ and set

$$R_m = R[\zeta_m, t^{1/m}] \text{ and } S_m = R_m\left[\frac{1}{2}, \sqrt{\delta}\right]$$

we can show the following theorem.

Theorem 12.1. Let $f_a(t,q)$ be a Nahm sum with power series term $g_{a,m}(t,q)$ as $q \to \zeta_m$. Then

$$g_{a,m}(t^{1/m}, q) \in \frac{\sqrt{\delta}}{\sqrt[m]{\varepsilon_m}} R_m\left[\frac{1}{m}\right][[\varepsilon]] \subseteq \frac{\sqrt{\delta}}{\sqrt[m]{\varepsilon_m}} S_m\left[\frac{1}{m}\right][[\varepsilon]]$$

where $\delta = z + a(1-z)$ and $\varepsilon_m \in H^1(R_m, \mu_m) \cong R_m^\times/(R_m^\times)^m$ is the modulo m regulator of the relative motivic cohomology class $V^{\mathrm{univ}} \in H^1\left(\mathbb{Z}(2)(R/\mathbb{Z}[t])\right)$.

Moreover, each μ_m-torsor naturally gives rise to a $\mathbb{G}_m$-torsor which by canonical deformation over nilpotents gives rise to a line bundle L_m over $\mathrm{Spec}\left(R_m[\frac{1}{m}][[\varepsilon]]\right)$ and by change L'_m over $\mathrm{Spec}\left(S_m[\frac{1}{m}][[\varepsilon]]\right)$ for each ε_m. Sections of these line bundles are elements of the module $\frac{1}{\sqrt[m]{\varepsilon_m}} R_m[\frac{1}{m}][[\varepsilon]]$.

Geometrically, we want to consider restricting scalars from $(S_m \otimes \mathbb{Q})[[q-\zeta_m]] \cong (S_m \otimes \mathbb{Q})[[\varepsilon]]$ to $S_m[[q-\zeta_m]] \cong S_m[[\varepsilon]]$. We do so by considering the p-integral case comparing the expansions at $q=1$ and $q=\zeta_p$ and the ways in which they determine each other.

Fix a prime $p > 3$ and consider the embedding $\mathbb{Q} \to \mathbb{Q}_p$, where we have $\widehat{R} = R_p^\wedge, \widehat{S} = S_p^\wedge$ are the p-adic completions of R, S, respectively. We obtain $\widehat{R_m} = \widehat{R}\widehat{\otimes}_{\mathbb{Z}_p\langle t\rangle}\mathbb{Z}_p\langle\zeta_{p^m}, t^{1/p^m}\rangle, \widehat{S_m} = \widehat{S}\widehat{\otimes}_{\mathbb{Z}_p\langle t\rangle}\mathbb{Z}_p\langle\zeta_{p^m}, t^{1/m}\rangle$ analogously. Passing to the limit, we have

$$\widehat{R_\infty} = \lim_m \widehat{R_m}$$

which is an integral perfectoid ring over $R^0_\infty = \mathbb{Z}_p\langle\zeta_{p^\infty}, t^{1/p^\infty}\rangle$. In this setting, we can consider the tilt $R^{0,\flat}_\infty = \lim_{x\mapsto x^p}(\mathbb{Z}_p\langle\zeta_p, t^{1/p^m}\rangle/p), \widehat{R_\infty}^\flat = \lim_{x\mapsto x^p}(\widehat{R_\infty}/p)$.

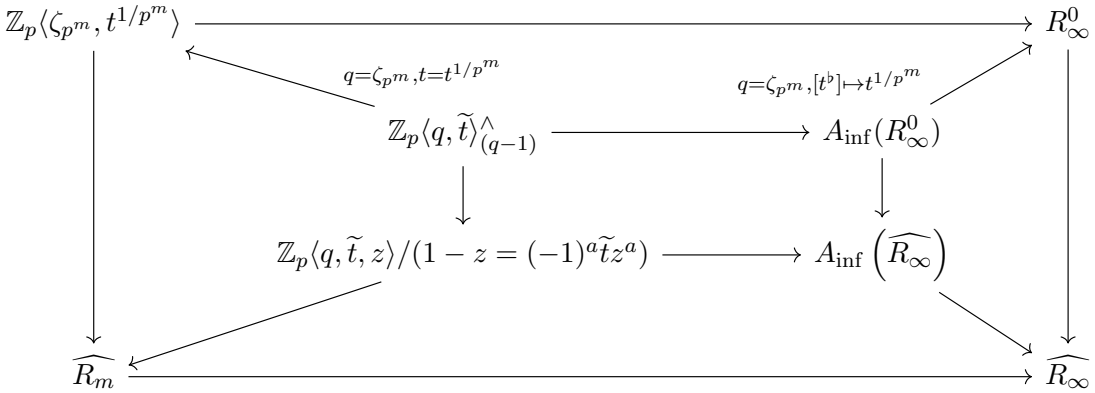
Example 12.2. Consider $\mathbb{Z}_p\langle\zeta_{1/p^\infty}\rangle$. The tilt $\mathbb{Z}\langle\zeta_{p^\infty}\rangle^\flat$ contains the element $(\overline{1}, \overline{\zeta_p}, \overline{\zeta_{p^2}}, \dots) = \varepsilon$ and we can produce an isomorphism $\mathbb{Z}\langle\zeta_{p^\infty}\rangle^\flat \cong \mathbb{F}_p[[(\varepsilon-1)^{1/p^\infty}]]$.

Per the previous example, there is a map $\mathbb{Z}_p\langle\zeta_{p^\infty}\rangle^\flat \mapsto R_\infty^{0,\flat}$ taking the element ε to $(t, t^{1/p}, t^{1/p^2}, \dots)$ which we denote $t^\flat$.

In p-adic Hodge theory, we also have the $A_{\mathrm{inf}}(-)$ construction taking a ring to the ring of its p-typical Witt vectors. We have $A_{\mathrm{inf}}(\mathbb{Z}_p\langle\zeta_{p^\infty}\rangle) \cong \mathbb{Z}_p\langle q^{1/p^\infty}\rangle^\wedge_{(q-1)}$, $A_{\mathrm{inf}}(R_\infty^0) \cong \mathbb{Z}_p\langle q^{1/p^\infty}, [t^\flat]^{1/p^\infty}\rangle^\wedge_{(q-1)}$. These rings are still defined by the Nahm equation, replacing t by the Teichmüller lift $[t^\flat]$ of t. This gives the pushout diagram

$$\begin{array}{ccc} \mathbb{Z}_p\langle q, \widetilde{t}\rangle^\wedge_{(q-1)} & \longrightarrow & A_{\mathrm{inf}}(R_\infty^0) \\ \downarrow & & \downarrow \\ \mathbb{Z}_p\langle q, \widetilde{t}, z\rangle/(1 - z = (-1)^a \widetilde{t} z^a) & \longrightarrow & A_{\mathrm{inf}}\left(\widehat{R_\infty}\right) \end{array}$$

and where the map $\mathbb{Z}_p\langle q, \widetilde{t}\rangle^\wedge_{(q-1)} \to A_{\mathrm{inf}}(R_\infty^0)$ is by $\widetilde{t} \mapsto [t^\flat]$. On specializations this diagram gives rise to the commutative cube



where the map $\widehat{R_m} \to \widehat{R_\infty}$ takes z to a Frobenius twist thereof.

In fact, there is a line bundle on $\mathbb{Z}_p\langle q, \widetilde{t}, z\rangle/(1 - z = (-1)^a \widetilde{t} z^a)$ given by

interpolating the line bundles L_m over $\widehat{R_m}[\frac{1}{p}][[q - \zeta_{p^m}]]$ such that the Nahm sum is a section of L_m for each m.

Proposition 12.3. The set of all $g_a(t, q) \in \widehat{R}[\frac{1}{p}][[q-1]]$ with constant coefficient 1 at $q = 1$ such that

$$\log\left(g_{a,1}(t^p, q^p)\right) - p \cdot \log\left(g_{a,1}(t, q)\right) \in -\frac{V(t^p) - p^2 \cdot V(t)}{p \log(q)} + \frac{p}{q-1}\widehat{R}[[q-1]]$$

defines a line bundle L_1 over $\widehat{R}[[q-1]]$ that is canonically trivialized under base change to $\widehat{R}[\frac{1}{p}][[q-1]]$.

Theorem 12.4. $g_{a,1}(t, q)$ is an element of $\sqrt{\delta}\widehat{R}[\frac{1}{p}][[q-1]]$.

Proof. This follows from the admissability of $f_a(t, q)$ as a series as in Theorem 5.6. ∎

Theorem 12.5. The base change of the line bundle L_1 of Proposition 12.3 along the map $\widehat{R}[[q-1]] \to \widehat{R_m}[\frac{1}{p}][[q-\zeta_{p^m}]]$ recovers the line bundle L_m.

This suggests a p-adic compatibility of the de Rham and étale realizations of V^{univ}.

Theorem 12.6. Mapping $g_{a,1}(t,q)$ under this comparison of line bundles one recovers the asymptotics of $g_{a,m}(t^{1/m},q)$.

Aspirationally, for any scheme S we would expect a theory of S-families of shtukas which are vector bundles over a complicated (analytic) stack $\mathrm{Spec}(\mathbb{Z}) \times S$ and where motives over S are examples of such shtukas. These line bundles that q-series-like Nahm sums ought be sections of live over $S \times \mathrm{Spec}(\mathbb{Z})$. Snappy.

Appendix A. The Spectral Sequence Computation of Lecture 8

This appendix contains a number of diagrams for the spectral sequence computation presented in Section 8.

$$
\begin{array}{ccccccc}
H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^4_{\neq}]_{\Sigma_4}\right)\otimes\mathbb{Q} & \longrightarrow & H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^3_{\neq}]_{\Sigma_3}\right)\otimes\mathbb{Q} & \longrightarrow & H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^2_{\neq}]_{\Sigma_2}\right)\otimes\mathbb{Q} & \longrightarrow & H_0\left(\mathrm{GL}_2(F), \mathbb{Z}[\mathbb{P}^1(F)]\right)\otimes\mathbb{Q} \\
H_1\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^3_{\neq}]_{\Sigma_3}\right)\otimes\mathbb{Q} & \longrightarrow & H_1\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^2_{\neq}]_{\Sigma_2}\right)\otimes\mathbb{Q} & \longrightarrow & H_1\left(\mathrm{GL}_2(F), \mathbb{Z}[\mathbb{P}^1(F)]\right)\otimes\mathbb{Q} & & \\
H_2\left(\mathrm{GL}_2(F), \mathbb{Z}[(\mathbb{P}^1(F))^2_{\neq}]_{\Sigma_2}\right)\otimes\mathbb{Q} & \longrightarrow & H_2\left(\mathrm{GL}_2(F), \mathbb{Z}[\mathbb{P}^1(F)]\right)\otimes\mathbb{Q} & & & & \\
H_3\left(\mathrm{GL}_2(F), \mathbb{Z}[\mathbb{P}^1(F)]\right)\otimes\mathbb{Q} & & & & & &
\end{array}
\tag{A.1}
$$

$$
\begin{array}{ccccc}
\mathbb{Q}[F^\times\setminus\{1\}]_{\Sigma_4} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & 0 & \longrightarrow & F^\times\otimes\mathbb{Q} \\
 & & & & \bigwedge^2((F^\times)^2\otimes\mathbb{Q})_{\Sigma_2} \\
 & & & & \bigwedge^3((F^\times)^2\otimes\mathbb{Q})
\end{array}
\longrightarrow \mathbb{Q}
\tag{A.2}
$$

References

[OEISa] *A003114: Number of partitions of n into parts 5k+1 or 5k+4*. URL: https://oeis.org/A003114.

[AKN24] Benjamin Antieau, Achim Krause, and Thomas Nikolaus. *Prismatic cohomology relative to δ-rings*. 2023. arXiv: 2310.12770 [math.AG]. URL: https://arxiv.org/abs/2310.12770.

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". English. In: *Ann. Math. (2)* 196.3 (2022), pp. 1135–1275. ISSN: 0003-486X. DOI: 10.4007/annals.2022.196.3.5.

[Bou24] Tess Bouis. *Motivic cohomology of mixed characteristic schemes*. 2024. arXiv: 2412.06635 [math.AG]. URL: https://arxiv.org/abs/2412.06635.

[CGZ21] Frank Calegari, Stavros Garoufalidis, and Don Zagier. *Bloch groups, algebraic K-theory, units, and Nahm's Conjecture*. 2021. arXiv: 1712.04887 [math.NT]. URL: https://arxiv.org/abs/1712.04887.

[FG05] Eric M. Friedlander and Daniel R. Grayson, eds. *Handbook of K-theory. Vol. 1 and 2*. English. Berlin: Springer, 2005. ISBN: 3-540-23019-X. DOI: 10.1007/978-3-540-27855-9.

[FS02] Eric M. Friedlander and Andrei Suslin. "The spectral sequence relating algebraic K-theory to motivic cohomology". English. In: *Ann. Sci. Éc. Norm. Supér. (4)* 35.6 (2002), pp. 773–875. ISSN: 0012-9593. DOI: 10.1016/S0012-9593(02)01109-6. URL: https://eudml.org/doc/82590.

[GZ21] Stavros Garoufalidis and Don Zagier. "Asymptotics of Nahm sums at roots of unity". English. In: *Ramanujan J.* 55.1 (2021), pp. 219–238. ISSN: 1382-4090. DOI: 10.1007/s11139-020-00266-x.

[GS+24] Stavros Garoufalidis et al. *The Habiro ring of a number field*. 2024. arXiv: 2412.04241 [math.NT]. URL: https://arxiv.org/abs/2412.04241.

[Hab04] Kazuo Habiro. "Cyclotomic completions of polynomial rings". English. In: *Publ. Res. Inst. Math. Sci.* 40.4 (2004), pp. 1127–1146. ISSN: 0034-5318. DOI: 10.2977/prims/1145475444.

[HW21] Fabian Hebestreit and Ferdinand Wagner. *Lecture Notes for Algebraic and Hermetian K-Theory*. 2021. URL: https://florianadler.github.io/AlgebraBonn/KTheory.pdf.

[HM24] Kaif Hilman and Jonas McCandless. *Lecture Notes on Algebraic K-Theory*. 2024. URL: https://sites.google.com/view/jonasmccandless/introduction-to-algebraic-k-theory.

[KS11] Maxim Kontsevich and Yan Soibelman. "Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants". English. In: *Commun. Number Theory Phys.* 5.2 (2011), pp. 231–352. ISSN: 1931-4523. DOI: 10.4310/CNTP.2011.v5.n2.a1.

[MW24] Samuel Meyer and Ferdinand Wagner. *Derived q-Hodge complexes and refined* TC^-. 2024. arXiv: 2410.23115 [math.AT]. URL: https://arxiv.org/abs/2410.23115.

[M-CFT] J.S. Milne. *Class Field Theory (v4.03)*. Available at www.jmilne.org/math/. 2020.

[M-ANT] James S. Milne. *Algebraic Number Theory (v3.08)*. Available at www.jmilne.org/math/. 2020.

[Sch17] Peter Scholze. "Canonical q-deformations in arithmetic geometry". English. In: *Ann. Fac. Sci. Toulouse, Math. (6)* 26.5 (2017), pp. 1163–1192. ISSN: 0240-2963. DOI: 10.5802/afst.1563.

[Stacks] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2023.

[Wag24] Ferdinand Wagner. *q-Witt vectors and q-Hodge complexes*. 2024. arXiv: 2410.23078 [math.NT]. URL: https://arxiv.org/abs/2410.23078.

[Wag22] Ferdinand Wagner. *On q-de Rham cohomology*. MS thesis at the Universität Bonn. 2022.

[Wei13] Charles A. Weibel. *The K-book. An introduction to algebraic K-theory*. English. Vol. 145. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2013. ISBN: 978-0-8218-9132-2.

[Wei02b] Eric W. Weisstein. *Bernoulli Polynomial*. 2002. URL: https://mathworld.wolfram.com/BernoulliPolynomial.html.

[Wei02a] Eric W. Weisstein. *Polylogarithm*. 2002. URL: https://mathworld.wolfram.com/Polylogarithm.html.

[Zag07] Don Zagier. *Frontiers in number theory, physics, and geometry II. On conformal field theories, discrete groups and renormalization. Papers from the meeting, Les Houches, France, March 9–21, 2003*. English. Ed. by Pierre Cartier et al. Berlin: Springer, 2007. Chap. The Dilogarithm Function. ISBN: 978-3-540-30307-7. DOI: 10.1007/978-3-540-30308-4.

Universität Bonn, Bonn, D-53113
Email address: wgabrielong@uni-bonn.de
URL: https://wgabrielong.github.io/